

# TOROIDALIZATION OF GENERATING SEQUENCES IN DIMENSION TWO FUNCTION FIELDS

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ABSTRACT. Let  $\mathbf{k}$  be an algebraically closed field of characteristic 0, and let  $K^*/K$  be a finite extension of algebraic function fields of transcendence degree 2 over  $\mathbf{k}$ . Let  $\nu^*$  be a  $\mathbf{k}$ -valuation of  $K^*$  with valuation ring  $V^*$ , and let  $\nu$  be the restriction of  $\nu^*$  to  $K$ . Suppose that  $R \rightarrow S$  is an extension of algebraic regular local rings with quotient fields  $K$  and  $K^*$  respectively, such that  $V^*$  dominates  $S$  and  $S$  dominates  $R$ . We prove that there exist sequences of quadratic transforms  $R \rightarrow \bar{R}$  and  $S \rightarrow \bar{S}$  along  $\nu^*$  such that  $\bar{S}$  dominates  $\bar{R}$  and the map between generating sequences of  $\nu$  and  $\nu^*$  has a toroidal structure. Our result extends the Strong Monomialization theorem of Cutkosky and Piltant.

## 1. INTRODUCTION

Let  $\mathbf{k}$  be an algebraically closed field of characteristic 0, and let  $K$  be an algebraic function field over  $\mathbf{k}$ . Throughout this paper we say that a subring  $R$  of  $K$  is *algebraic* if  $R$  is essentially of finite type over  $\mathbf{k}$ . We will denote the maximal ideal of a local ring  $R$  by  $m_R$ .

Let  $K^*/K$  be a finite extension of algebraic function fields over  $\mathbf{k}$ . Let  $\nu^*$  be a  $\mathbf{k}$ -valuation of  $K^*$  with valuation ring  $V^*$  and value group  $\Gamma^*$ . Let  $\nu$  be the restriction of  $\nu^*$  to  $K$  with valuation ring  $V$  and value group  $\Gamma$ . Consider an extension of algebraic regular local rings  $R \rightarrow S$  where  $R$  has quotient field  $K$ ,  $S$  has quotient field  $K^*$ ,  $R$  is dominated by  $S$  and  $S$  is dominated by  $V^*$  (i.e.,  $m_V \cap R = m_R$  and  $m_{V^*} \cap S = m_S$ ).

It has been a classical topic to investigate finite extensions of rings of algebraic integers and mappings between algebraic curves. In these cases  $K$  and  $K^*$  are of transcendence degree 1 over  $\mathbf{k}$ , and the homomorphisms of local rings of points are ramified maps  $R \rightarrow S$  of discrete (rank 1) valuation rings. We have that  $R = V$  and  $S = V^*$  are local Dedekind domains. Suppose that  $(u) = m_R$  and  $(x) = m_S$  are the maximal ideals of  $R$  and  $S$ , respectively, then

$$u = x^e \delta \tag{1.1}$$

where  $\delta \in S$  is a unit. The corresponding value groups are  $\Gamma \cong \mathbb{Z}$  and  $\Gamma^* \cong \mathbb{Z}$  and we have a natural isomorphism  $\Gamma^*/\Gamma \cong \mathbb{Z}_e$ .

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The study of ramification theory, in general, for valuations in arbitrary fields was initiated by Krull and further pursued by many authors (cf. [5] and the literature cited there).

In this paper, we are interested in valuations of two dimensional algebraic function fields, i.e., the situation when  $K$  and  $K^*$  are of transcendence degree 2 over  $\mathbf{k}$ . Valuations in dimension two are completely described by a compact set of data called *generating sequence*. Generating sequences provide a very useful tool in the study of algebraic surfaces (cf. [2, 5, 8, 7, 10, 11]).

We shall briefly recall the definition of generating sequences, as in [10]. Let  $\Gamma_+ = \nu(R \setminus \{0\})$  be the semigroup of  $\Gamma$  consisting of the values of nonzero elements of  $R$ . For  $\gamma \in \Gamma_+$ , let  $I_\gamma = \{f \in R \mid \nu(f) \geq \gamma\}$ . A (possibly infinite) sequence  $\{Q_i\}$  of elements of  $R$  is a generating sequence for  $\nu$  if for every  $\gamma \in \Gamma_+$

$$I_\gamma = \left\{ \prod_i Q_i^{a_i} \mid a_i \in \mathbb{N}_0, \sum_i a_i \nu(Q_i) \geq \gamma \right\}.$$

A generating sequence of  $\nu$  is *minimal* if none of its proper subsequences is a generating sequence for  $\nu$ . If  $\{Q_i\}$  is a minimal generating sequence then  $\{\nu(Q_i)\} \subset \Gamma$  forms a minimal set of generators for  $\Gamma_+$ .

Notice that a generating sequence  $\{Q_i\}$  in  $R$  and the values  $\{\nu(Q_i)\}$  completely describe the valuation  $\nu$  (see [10, Definition 1.1] and [5, Section 7.5] for more detailed discussions). A generating sequence of  $\nu^*$  in  $S$  can be defined similarly.

The aim of this paper is to find structure theorems for generating sequences of  $\nu$  and  $\nu^*$ . Our work is inspired by the Strong Monomialization theorem of Cutkosky and Piltant [5, Theorem 4.8], which we recall below.

We first need few definitions. Suppose that  $R$  is a local domain. A *monoidal* transform  $R \rightarrow R'$  is a birational extension of local domains such that  $R' = R\left[\frac{I}{x}\right]_m$  where  $I$  is a regular prime ideal of  $R$ ,  $0 \neq x \in I$  and  $m$  is a prime ideal of  $R\left[\frac{I}{x}\right]$  such that  $m \cap R = m_R$ . If  $I = m_R$  then the monoidal transform  $R \rightarrow R'$  is called a *quadratic* transform. In our situation (dimension two) since  $R$  is a regular local ring, any nontrivial monoidal transform  $R \rightarrow R'$  is a quadratic transform and there exists a regular system of parameters  $(u, v)$  of  $R$  such that  $R' = R\left[\frac{u}{v}\right]_m$ , where  $m$  is a maximal ideal of  $R\left[\frac{u}{v}\right]$ . We say that  $R \rightarrow R'$  is a monoidal transform *along*  $\nu$  if  $\nu$  dominates  $R'$ .

The celebrated Local Monomialization theorem of Cutkosky [3, Theorem 1.1] states that there exist sequences of monoidal transforms  $R \rightarrow R_1$  and  $S \rightarrow S_1$  such that  $\nu^*$  dominates  $S_1$ ,  $S_1$  dominates  $R_1$ , and there are regular parameters  $(u, v)$  in  $R_1$  and  $(x, y)$  in  $S_1$ , units  $\delta_1, \delta_2 \in S_1$  and a matrix  $A = (a_{ij})$  of nonnegative integers such that  $\det A \neq 0$  and

$$\begin{cases} u &= x^{a_{11}} y^{a_{12}} \delta_1 \\ v &= x^{a_{21}} y^{a_{22}} \delta_2. \end{cases} \quad (1.2)$$

The existence of  $R_1$  and  $S_1$  such that (1.2) holds follows directly from the standard theorems on resolution of singularities, but in general we will not have the essential condition that  $\det A \neq 0$ . The difficulty in Cutkosky's work is to achieve the condition  $\det A \neq 0$  (we should note that Cutkosky's Local Monomialization theorem is valid in arbitrary dimension).

In our situation, under the additional assumption that  $\Gamma^*$  is a non-discrete subgroup of  $\mathbb{Q}$  (which is the essential and subtle case), the Strong Monomialization theorem of Cutkosky and Piltant [5, Theorem 4.8] further assures that  $A$  can be taken to have the following special form

$$A = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}. \quad (1.3)$$

Strong Monomialization is an important and useful result. It shows that no matter how complicated the structure of the extension  $R \subset S$  is, after blowing-up we obtain a simultaneous resolution, that is, an extension  $R_1 \subset S_1$  of regular local rings such that  $S_1$  is the localization of the integral closure of  $R_1$  in  $K^*$ , and this extension is very nice and simple, since it is monomial.

Observe that  $u, v \in R_1$  (resp.  $x, y \in S_1$ ) are the first two members of a generating sequence of  $\nu$  (resp.  $\nu^*$ ). Therefore, (1.3) exhibits a toroidal structure of the first two elements of such generating sequences.

The goal of our work is to investigate toroidal structures of generating sequences of  $\nu$  and  $\nu^*$ . Our main theorem is stated as follows.

**Theorem 1.1** (Theorem 8.1). *Let  $\mathbf{k}$  be an algebraically closed field of characteristic 0, and let  $K^*/K$  be a finite extension of algebraic function fields of transcendence degree 2 over  $\mathbf{k}$ . Let  $\nu^*$  be a  $\mathbf{k}$ -valuation of  $K^*$  with valuation ring  $V^*$ , and let  $\nu$  be the restriction of  $\nu^*$  to  $K$ . Suppose that  $R \rightarrow S$  is an extension of algebraic regular local rings with quotient fields  $K$  and  $K^*$  respectively, such that  $V^*$  dominates  $S$  and  $S$  dominates  $R$ . Then there exist sequences of quadratic transforms  $R \rightarrow \bar{R}$  and  $S \rightarrow \bar{S}$  along  $\nu^*$  such that  $\bar{S}$  dominates  $\bar{R}$  and the map between generating sequences of  $\nu$  and  $\nu^*$  in  $\bar{R}$  and  $\bar{S}$  respectively, has a toroidal structure.*

The precise toroidal structures of generating sequences of  $\nu$  and  $\nu^*$  are described in details in Section 2.

To prove our main theorem, we consider different cases according to Zariski's classification of valuations in two dimensional function fields over an algebraically closed field of characteristic 0 [12]. In most cases, the result follows from a standard application of the Strong Monomialization theorem of Cutkosky and Piltant. These cases are analyzed in Section 3. The bulk of the paper is devoted to the essential case, when  $\Gamma^*$  is a non-discrete subgroup of  $\mathbb{Q}$ . We shall now briefly describe the main steps of the proof in this case.

Let  $(x, y)$  be a regular system of parameters in  $S$ . We begin in Section 4 by constructing a sequence of *jumping polynomials*  $\{T_i\}_{i \geq 0}$  in  $S$  with  $T_0 = x$  and  $T_1 = y$ ,

which contains as a subsequence a minimal generating sequence of  $\nu^*$ . Our notion of jumping polynomials is very similar to Favre and Jonsson's notion of *key polynomials* [7]. The idea of key polynomials is originally due to MacLane [9]. By normalizing, we may assume that  $\nu^*(x) = 1$ . Let  $\nu^*(y) = \frac{p_1}{q_1}$ , where  $p_1$  and  $q_1$  are coprime positive integers. For each  $i \geq 1$ , we define  $T_{i+1}$  recursively. Let  $p_{i+1}$  and  $q_{i+1}$  be coprime positive integers defined by

$$\nu^*(T_{i+1}) = q_i \nu^*(T_i) + \frac{1}{q_1 \cdots q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

Our proof of Theorem 1.1 proceeds in the following line of arguments.

- (1) We observe that the collection of jumping polynomials  $\{T_i\}_{i \geq 0}$  forms a generating sequence of  $\nu^*$  in  $S$  (Theorem 4.6).
- (2) Let  $d = d(R, S)$  be the greatest common divisor of  $\{p_i \mid i \geq 1\}$ . We show that the powers of  $x$  that appear in  $T_i$  are multiple of  $d$  for all  $i \geq 2$  (Corollary 7.2). In other words  $T_i$ , for  $i \geq 2$ , is a polynomial in  $x^d$  and  $y$ .
- (3) Let us denote by  $t(R, S)$  the power  $t$  in (1.3) obtained from the Strong Monomialization theorem. For simplicity, assume that the constant  $\delta_1$  of (1.2) is equal to 1. We observe that if  $t(R, S)$  divides  $d$  then  $T_i$ , for  $i \geq 2$ , is a polynomial in  $u$  and  $v$ . This shows that  $u$ , together with the collection  $\{T_i\}_{i \geq 1}$ , form a generating sequence of  $\nu$  in  $R$ . Therefore we obtain the desired toroidal structure.
- (4) The core of our argument is to show that if  $t(R, S)$  does not divide  $d(R, S)$  then we can find sequences of quadratic transforms  $R \rightarrow R'$  and  $S \rightarrow S'$  so that  $t(R', S') < t(R, S)$ . More precisely, let  $M = \min\{i > 0 \mid t \nmid p_i\}$ . Then  $t(R', S')$  is the greatest common divisor of  $t$  and  $p_M$ . Lemma 8.2 is crucial in the proof of this step.

Finally, starting with a sequence of jumping polynomials in  $S'$  and repeating the above process, after a finite number of iterations we end up with the situation where  $t(R', S')$  divides  $d(R', S')$ . Then we conclude as in (3).

We remark that in order to make our arguments work we need a very explicit description of the quadratic transforms that we perform. There are several preparatory lemmas to this end.

## 2. STATEMENT OF THE RESULT

Let  $\mathbf{k}$  be an algebraically closed field of characteristic 0 and let  $K^*/K$  be a finite extension of algebraic function fields of transcendence degree 2 over  $\mathbf{k}$ . Let  $\nu^*$  be a  $\mathbf{k}$ -valuation of  $K^*$  with valuation ring  $V^*$  and value group  $\Gamma^*$  and let  $\nu$  be the restriction of  $\nu^*$  to  $K$  with valuation ring  $V$  and value group  $\Gamma$ .

Suppose that  $S$  is an algebraic regular local ring with quotient field  $K^*$  which is dominated by  $V^*$  and  $R$  is an algebraic regular local ring with quotient field  $K$  which is dominated by  $S$ . We will show that there exist sequences of quadratic transforms

$R \rightarrow R'$  and  $S \rightarrow S'$  along  $\nu^*$  such that  $S'$  dominates  $R'$  and the map between generating sequences of  $S'$  and  $R'$  has the following toroidal structure.

- (1) If  $\nu^*$  is divisorial then  $R' = V$  and  $S' = V^*$  with regular parameters  $u \in R'$  and  $x \in S'$  such that  $u = x^a \gamma$  for some unit  $\gamma \in S'$ .
- (2) If  $\nu^*$  has rank 2 then there exist regular parameters  $(x, y)$  in  $S'$  and  $(u, v)$  in  $R'$  such that  $\{x, y\}$  is a minimal generating sequence for  $\nu^*$ ,  $\{u, v\}$  is a minimal generating sequence for  $\nu$  and

$$\begin{aligned} u &= x^a y^b \delta \\ v &= y^d \gamma \end{aligned}$$

for some units  $\delta, \gamma \in S'$ , and for some nonnegative integers  $a, b, d$  such that  $ad \neq 0$ .

- (3) If  $\nu^*$  has rank 1 and rational rank 2 then there exist regular parameters  $(x, y)$  in  $S'$  and  $(u, v)$  in  $R'$  such that  $\{x, y\}$  is a minimal generating sequence for  $\nu^*$ ,  $\{u, v\}$  is a minimal generating sequence for  $\nu$  and

$$\begin{aligned} u &= x^a y^b \delta \\ v &= x^c y^d \gamma \end{aligned}$$

for some units  $\delta, \gamma \in S'$ , and for some nonnegative integers  $a, b, c, d$  such that  $ad - bc \neq 0$ .

- (4) If  $\Gamma$  and  $\Gamma^*$  are non-discrete subgroups of  $\mathbb{Q}$  then there exist a generating sequence  $\{H_i\}_{i \geq 0}$  of  $\nu^*$  in  $S'$  and regular parameters  $(u, v)$  in  $R'$  such that

$$\begin{aligned} u &= H_0^a \gamma \\ v &= H_1 \end{aligned}$$

for some unit  $\gamma \in S'$ , and  $H_i \in R'$  for all  $i > 1$ . In particular,  $\{u, \{H_i\}_{i > 0}\}$  is a generating sequence of  $\nu$  in  $R'$ . Furthermore, if  $\nu^*(H_0)$  is not a multiple of  $\nu^*(H_1)$  then  $\{H_i\}_{i \geq 0}$  is a minimal generating sequence of  $\nu^*$  in  $S'$ . If  $\nu^*(H_0)$  is a multiple of  $\nu^*(H_1)$  then  $\{H_i\}_{i \geq 1}$  is a minimal generating sequence of  $\nu^*$  in  $S'$ .

- (5) If  $\nu$  is discrete but not divisorial then there exist regular parameters  $(x, y)$  in  $S'$  and  $(u, v)$  in  $R'$  such that  $\Gamma^*$  is generated by  $\nu^*(x)$ ,  $\Gamma$  is generated by  $\nu(u)$  and  $u = x^a \gamma$  for some unit  $\gamma \in S'$ . Moreover,  $S'$  has a non-minimal generating sequence  $\{x, \{T_i\}_{i > 0}\}$  such that  $\{u, \{T_i\}_{i > 0}\}$  form a non-minimal generating sequence in  $R'$ .

### 3. VALUATIONS IN 2 DIMENSIONAL FUNCTION FIELDS

Zariski in [12] gave a classification of valuations in two dimensional function fields over an algebraically closed field of characteristic zero. We refer to [6] (Chapter 8, Section 1) for a modern treatment of the subject and for the definitions and background needed in this section.

We will prove our main theorem by analyzing the different types of valuations of  $K^*$ . Notations are as in Section 2.

**3.1. One dimensional valuations.** By definition,  $\nu^*$  is divisorial. In this case  $\nu$  and  $\nu^*$  are discrete, and  $V$  and  $V^*$  are iterated quadratic transforms of  $R$  and  $S$  respectively (see [1, Proposition 4.4]).

Let  $u$  be a regular parameter of  $V$  and let  $x$  be a regular parameter of  $V^*$ . Then there is a relation

$$u = x^a \gamma$$

where  $\gamma \in V^*$  is a unit, and  $a \geq 1$ . Since  $\{u\}$  is a minimal generating sequence for  $V$ , and  $\{x\}$  is a minimal generating sequence for  $V^*$  the theorem is proved.

**3.2. Zero dimensional valuations of rational rank 2.** By Strong Monomialization [5, Theorem 4.8] there exist sequences of quadratic transforms  $R \rightarrow R'$  and  $S \rightarrow S'$  along  $\nu^*$  such that  $R'$  has regular parameters  $(u, v)$ ,  $S'$  has regular parameters  $(x, y)$ , and

$$\begin{aligned} u &= x^a y^b \delta \\ v &= x^c y^d \gamma \end{aligned}$$

for some units  $\delta, \gamma \in S'$  and for some nonnegative integers  $a, b, c, d$  such that  $ad - bc \neq 0$ . Further,  $c = 0$  if  $\nu^*$  has rank two. We also have that  $\{\nu^*(x), \nu^*(y)\}$  is a rational basis of  $\Gamma^* \otimes \mathbb{Q}$ , and  $\{\nu(u), \nu(v)\}$  is a rational basis of  $\Gamma \otimes \mathbb{Q}$ .

Let  $z \in S'$ . Then  $z \in \hat{S}' = S'/m_{S'}[[x, y]] = \mathbf{k}[[x, y]]$ , since  $\nu^*$  is zero dimensional and  $\mathbf{k}$  is algebraically closed. Observe that  $z$  has an expansion  $z = \sum_{i \geq 1} a_i x^{b_i} y^{c_i}$ , where  $a_i \in \mathbf{k}$ ,  $b_i$  and  $c_i$  are non negative integers, and the terms have increasing value, since  $\nu^*(x)$  and  $\nu^*(y)$  are rationally independent. It follows that  $\nu^*(z) = b_1 \nu^*(x) + c_1 \nu^*(y)$ . Hence  $\{x, y\}$  is a minimal generating sequence of  $\nu^*$  in  $S'$ , and similarly  $\{u, v\}$  is a minimal generating sequence of  $\nu$  in  $R'$ , and the theorem is proved.

The rest of the paper will be devoted to studying the remaining cases, that is zero dimensional valuations of rational rank 1.

**3.3. Non-discrete zero dimensional valuations of rational rank 1.** We can normalize  $\Gamma^*$  so that it is an ordered subgroup of  $\mathbb{Q}$ , whose denominators are not bounded, as  $\Gamma^*$  is not discrete. In Example 3, Section 15, Chapter VI of [13], examples are given of two-dimensional algebraic function fields with value group equal to any given subgroup of the rational numbers. This case is much more subtle.

**3.4. Discrete zero dimensional valuations of rational rank 1.** If  $\nu^*$  is discrete, then  $\nu$  is also discrete. This case will be handled in the same way as the case of non-discrete zero dimensional valuations of rational rank 1, but the generating sequences of  $\nu^*$  and  $\nu$  will not be minimal.

## 4. CONSTRUCTION OF JUMPING POLYNOMIALS

From now on we work under the assumption that the value group of  $\nu^*$  is a subgroup of  $\mathbb{Q}$  and  $\text{trdeg}_{\mathbf{k}}(V^*/m_{V^*}) = 0$ . Let  $(x, y)$  be a system of regular parameters in  $S$ . We normalize the value group  $\Gamma^*$  of  $K^*$  so that  $\nu^*(x) = 1$ .

We shall now construct a sequence of polynomials  $\{T_i\}_{i \geq 0}$  in  $S$ . Let

$$\begin{cases} T_0 &= x \\ T_1 &= y. \end{cases}$$

Set  $q_0 = \infty$  and choose a pair of coprime positive integers  $(p_1, q_1)$  so that  $\nu^*(y) = \frac{p_1}{q_1}$ . For  $i \geq 1$ ,  $T_{i+1}$  is defined recursively as follows. Let

$$T_{i+1} = T_i^{q_i} - \lambda_i \prod_{j=0}^{i-1} T_j^{n_{i,j}},$$

where  $n_{i,j} < q_j$  is a nonnegative integer such that  $q_i \nu^*(T_i) = \nu^*(\prod_{j=0}^{i-1} T_j^{n_{i,j}})$ , that is  $q_i \nu^*(T_i) = \sum_{j=0}^{i-1} n_{i,j} \nu^*(T_j)$ , and  $\lambda_i \in \mathbf{k}$  is the residue of  $\frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}}$ .

Finally, choose positive integers  $p_{i+1}$  and  $q_{i+1}$  so that  $(p_{i+1}, q_{i+1}) = 1$  and

$$\nu^*(T_{i+1}) = q_i \nu^*(T_i) + \frac{1}{q_1 \cdots q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

**Definition 4.1.** The polynomial  $T_i$  will be called the  $i$ -th *jumping polynomial* and the value  $\nu^*(T_i)$  will be called the  $i$ -th *j-value*. We will denote the  $i$ -th *j-value* by  $\beta_i$ . We say that  $\beta_i$  is an *independent j-value* if  $q_i \neq 1$ .

**Remark 4.2.** For  $i > 0$  let  $Q_i = q_1 \cdots q_i$ . Observe that  $Q_i \beta_i$  is an integer number,  $\beta_{i+1} = q_i \beta_i + \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}$  and  $q_{i+1} \beta_{i+1} \geq \beta_{i+1} > q_i \beta_i$ .

Consider the subsequence  $\{\beta_{i_l}\}_{l \geq 0}$  of all independent *j-values*. Let  $\bar{\beta}_l = \beta_{i_l}$  denote the  $l$ -th independent *j-value*,  $\bar{q}_l = q_{i_l}$  and  $H_l = T_{i_l}$ . Since

$$T_{i+1} = T_i^{q_i} - \lambda_i \prod_{j=0}^{i-1} T_j^{n_{i,j}}, \text{ where } 0 \leq n_{i,j} < q_j,$$

it follows that  $n_{i,j} = 0$  whenever  $q_j = 1$ . Therefore only the  $H_l$ 's with  $i_l < i$  will appear in the product  $\prod_{j=0}^{i-1} T_j^{n_{i,j}}$ . Thus, if  $0 < i = i_l$  then

$$T_{i+1} = H_l^{\bar{q}_l} - \lambda_i \prod_{j=0}^{l-1} H_j^{n_{i,i_j}}.$$

If  $i + 1 < i_{l+1}$  then  $q_{i+1} = 1$  and

$$T_{i+2} = T_{i+1} - \lambda_{i+1} \prod_{j=0}^l H_j^{n_{i+1}, i_j} = H_l^{\bar{q}_l} - \lambda_i \prod_{j=0}^{l-1} H_j^{n_{i+1}, i_j} - \lambda_{i+1} \prod_{j=0}^l H_j^{n_{i+1}, i_j}.$$

In general, the recursive formula for  $H_{l+1}$  with  $l > 0$  will be

$$\begin{aligned} H_{l+1} &= H_l^{\bar{q}_l} - \lambda_{i_l} \prod_{j=0}^{l-1} H_j^{n_{i_l}, i_j} - \lambda_{i_{l+1}} \prod_{j=0}^l H_j^{n_{i_{l+1}}, i_j} - \lambda_{i_{l+2}} \prod_{j=0}^l H_j^{n_{i_{l+2}}, i_j} - \dots \\ &\dots - \lambda_{i_{l+1}-1} \prod_{j=0}^l H_j^{n_{i_{l+1}-1}, i_j} = H_l^{\bar{q}_l} - \lambda_{i_l} \prod_{j=0}^{l-1} H_j^{n_{i_l}, i_j} - \sum_{i' = i_{l+1}}^{i_{l+1}-1} \lambda_{i'} \prod_{j=0}^l H_j^{n_{i'}, i_j}. \end{aligned}$$

We also notice that the sequence of independent jumping polynomials  $\{H_l\}_{l \geq 0}$  starts with  $H_0 = x$  and  $H_1 = y - \sum_{j=1}^{i_1-1} \lambda_j x^{\beta_j}$ .

Independent  $j$ -values furthermore have a number of basic properties. If  $l > 0$  and  $i_l \leq i < i_{l+1}$  then  $q_1 \cdots q_i = \bar{q}_1 \cdots \bar{q}_l$  and the following equalities hold

$$\begin{aligned} \bar{\beta}_0 &= \beta_0 = 1, \quad \bar{q}_0 = q_0 = \infty, \\ \bar{\beta}_1 &= p_1 + \cdots + p_{i_1-1} + \frac{p_{i_1}}{q_{i_1}}, \\ \bar{\beta}_{l+1} &= \bar{q}_l \bar{\beta}_l + \frac{p_{i_{l+1}} + p_{i_{l+2}} + \cdots + p_{i_{l+1}-1}}{\bar{q}_1 \cdots \bar{q}_l} + \frac{1}{\bar{q}_1 \cdots \bar{q}_l} \cdot \frac{p_{i_{l+1}}}{\bar{q}_{l+1}}. \end{aligned}$$

**Remark 4.3.** For all  $l > 0$  denote by  $\bar{Q}_l = \bar{q}_1 \cdots \bar{q}_l$  and  $\bar{p}_l = (p_{i_{l-1}+1} + \cdots + p_{i_l-1})\bar{q}_l + p_{i_l}$ . Then  $(\bar{p}_l, \bar{q}_l) = (p_{i_l}, q_{i_l}) = 1$ ,  $\bar{\beta}_1 = \frac{\bar{p}_1}{\bar{q}_1}$  and  $\bar{\beta}_{l+1} = \bar{q}_l \bar{\beta}_l + \frac{1}{\bar{Q}_l} \cdot \frac{\bar{p}_{l+1}}{\bar{q}_{l+1}}$ . In particular,  $\bar{q}_{l+1} \bar{\beta}_{l+1} > \bar{\beta}_{l+1} > \bar{q}_l \bar{\beta}_l$ .

**Remark 4.4.** In general, if  $(x, y)$  is a system of regular parameters in  $S$  we may not necessarily have  $\nu^*(x) = 1$ . Then in order to define a sequence of jumping polynomials  $\{T_i\}_{i \geq 0}$  such that  $T_0 = x$  and  $T_1 = y$ , we introduce the following valuation  $\tilde{\nu}$  of  $K^*$

$$\tilde{\nu}(f) = \frac{\nu^*(f)}{\nu^*(x)}$$

for all  $f \in K^*$ . Then  $\tilde{\nu}(x) = 1$  and we use the construction above with  $\nu^*$  replaced by  $\tilde{\nu}$ . This procedure is equivalent to normalizing the value group  $\Gamma^*$  so that  $\nu^*(x) = 1$ .

It is easy to see that the sequence of jumping polynomials  $\{T_i\}_{i \geq 0}$  in  $S$  is well defined (Corollary 5.10). The next goal is to show that it forms a generating sequence of  $\nu^*$ .

**4.1. Discrete case.** We suppose that the value group of  $\nu^*$  is isomorphic to  $\mathbb{Z}$ . Suppose also that  $\nu^*(x) = 1$  generates  $\Gamma^*$ . Then by [10] (p.154) we have that a

set  $\{Q_i\}_{i \geq 0} \subset S$  is a generating sequence of  $\nu^*$  if  $\nu^*(Q_0) = 1$ , each  $Q_i$  is a regular parameter of  $S$  such that  $(Q_0, Q_i)$  form a system of regular parameters, and  $\lim_{i \rightarrow \infty} \nu^*(Q_i) = \infty$ .

In particular, there are no minimal generating sequences in  $S$ . Any infinite subsequence of a generating sequence which contains  $Q_0$  is a generating sequence itself.

**Theorem 4.5.** *The above  $\{T_i\}_{i \geq 0} \subset S$  form a generating sequence of  $\nu^*$ .*

*Proof.* Since  $\nu^*(x)$  generates  $\Gamma^*$  we see that  $q_i = 1$  for all  $i \geq 1$ . Thus  $T_1 = y$  and  $T_{i+1} = y - \lambda_1 x^{n_{1,0}} - \lambda_2 x^{n_{2,0}} - \dots - \lambda_i x^{n_{i,0}}$  are linear in  $y$  for all  $i \geq 1$ . In particular  $T_i$  is a regular parameter of  $S$  and  $(x, T_i)$  form a system of regular parameters in  $S$ . Notice also that  $\beta_i = \nu^*(T_i) \in \mathbb{Z}$  and  $\beta_{i+1} > \beta_i$ . This implies that  $\lim_{i \rightarrow \infty} \nu^*(T_i) = \infty$ .  $\square$

**4.2. Non-discrete case.** We assume now that the value group of  $\nu^*$  is a non-discrete subgroup of  $\mathbb{Q}$ .

**Theorem 4.6.** *With notations as above,  $\{T_i\}_{i \geq 0} \subset S$  form a generating sequence of  $\nu^*$ . Furthermore, if  $p_1 > 1$  then  $\{H_l\}_{l \geq 0} \subset S$  form a minimal generating sequence of  $\nu^*$ . If  $p_1 = 1$  then  $\{H_l\}_{l \geq 1} \subset S$  form a minimal generating sequence of  $\nu^*$ .*

*Proof.* This is shown in [10], in Chapter 2 of [7], and in [2].  $\square$

We will give an alternative proof of the above theorem in Section 7.1. Namely, we will show that the set  $\{H_l\}_{l \geq 0} \subset S$  satisfies the definition of generating sequence given in [5].

## 5. ARITHMETICS

In this section we prove several properties of the numbers  $q_i$  and  $\beta_i$  defined in Section 4.

**Definition 5.1.** Given two rational numbers  $a$  and  $b$  we say that  $a$  is  $\mathbb{Z}$ -divisible by  $b$ , or equivalently  $b$   $\mathbb{Z}$ -divides  $a$ , and write  $b|a$ , if  $a$  is an integer multiple of  $b$ , that is  $a \in b\mathbb{Z}$  or  $a = nb$  for some  $n \in \mathbb{Z}$ . The *greatest common divisor* of  $a$  and  $b$ , denoted by  $(a, b)$ , is as usual the greatest rational number  $g$  such that  $g|a$  and  $g|b$ .

**Proposition 5.2.** *Let  $p, q, t$  be nonzero integers with  $(p, q) = 1$ . Then  $(\frac{1}{t}, \frac{p}{tq}) = \frac{1}{tq}$ .*

*Proof.* If  $g = (\frac{1}{t}, \frac{p}{tq})$  then  $\frac{1}{tq}|g$ . On the other hand

$$\frac{1}{tq} = \frac{1}{tq}(\alpha p + \beta q) = \alpha \frac{p}{tq} + \beta \frac{1}{t}$$

for some integers  $\alpha$  and  $\beta$  since  $1 = (p, q)$ . Thus  $g|_{\frac{1}{tq}}$ .  $\square$

**Proposition 5.3.** *For  $k \geq 1$ , we have  $(\beta_0, \beta_1, \dots, \beta_k) = \frac{1}{Q_k}$ .*

*Proof.* We use induction on  $k$ . For  $k = 1$ , clearly  $(\beta_0, \beta_1) = (1, \frac{p_1}{q_1}) = \frac{1}{q_1}$ . Assume now that  $(\beta_0, \beta_1, \dots, \beta_{k-1}) = \frac{1}{Q_{k-1}}$ . Then

$$(\beta_0, \beta_1, \dots, \beta_k) = ((\beta_0, \beta_1, \dots, \beta_{k-1}), \beta_k - q_{k-1}\beta_{k-1}) = \left(\frac{1}{Q_{k-1}}, \frac{p_k}{Q_{k-1}q_k}\right) = \frac{1}{Q_k}.$$

□

**Corollary 5.4.** *For  $k \geq 0$ , let  $\Gamma_k = \langle \beta_0, \beta_1, \dots, \beta_k \rangle$ . Then,  $\Gamma_k = \frac{1}{Q_k}\mathbb{Z}$  for all  $k \geq 1$ . (That is, the group generated by the values of the first  $k + 1$  jumping polynomials is isomorphic to  $\frac{1}{Q_k}\mathbb{Z}$ .)*

**Proposition 5.5.** *For  $k \geq 1$ , we have  $q_k\beta_k \in \Gamma_{k-1}$ . Moreover, if  $q_k > 1$  then  $q_k\beta_k$  has order  $q_k$  in  $\frac{\Gamma_{k-1}}{q_k\Gamma_{k-1}}$ .*

*Proof.* We have  $q_k\beta_k = q_kq_{k-1}\beta_{k-1} + p_k\frac{1}{Q_{k-1}}$  is  $\mathbb{Z}$ -divisible by  $\frac{1}{Q_{k-1}}$ . Thus,  $q_k\beta_k \in \Gamma_{k-1}$ .

Moreover, if  $q_k > 1$  then  $G = \frac{\Gamma_{k-1}}{q_k\Gamma_{k-1}} \cong \mathbb{Z}_{q_k}$  is not trivial and  $\text{ord}_G(q_k\beta_k) = \text{ord}_G(p_k\frac{1}{Q_{k-1}}) = \text{ord}_{\mathbb{Z}_{q_k}}(p_k) = q_k$  since  $(p_k, q_k) = 1$ . □

**Corollary 5.6.** *For  $k \geq 1$ , we have  $(\bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_k) = \frac{1}{Q_k}$  and  $\bar{\Gamma}_k = \langle \bar{\beta}_0, \bar{\beta}_1, \dots, \bar{\beta}_k \rangle = \frac{1}{Q_k}\mathbb{Z}$ . Also,  $\bar{q}_k\bar{\beta}_k \in \bar{\Gamma}_{k-1}$  and  $\bar{q}_k\bar{\beta}_k$  has order  $\bar{q}_k$  in  $\frac{\bar{\Gamma}_{k-1}}{\bar{q}_k\bar{\Gamma}_{k-1}}$ .*

**Remark 5.7.** With notations as above, we have  $\bar{\Gamma}_k = \Gamma_{i_k}$  and  $\bigcup_{k \geq 0} \bar{\Gamma}_k = \bigcup_{k \geq 0} \Gamma_k$ .

**Proposition 5.8.** *If  $x \in \Gamma_k$  and  $x \geq q_k\beta_k$  then there exists a unique representation*

$$x = \sum_{j=0}^k a_j\beta_j \quad (5.1)$$

*with integer coefficients  $0 \leq a_j < q_j$ .*

*Proof.* We first show existence of the presentation (5.1). We use induction on  $k$ . The claim is trivial for  $k = 0$ . Let  $x \in \Gamma_1$  and  $x \geq p_1$ , that is  $x = y + \frac{p}{q_1}$  for some  $p, y \in \mathbb{Z}$  such that  $y \geq p_1$  and  $0 \leq p < q_1$ . Since  $(p_1, q_1) = 1$  there exists an integer  $0 \leq a_1 < q_1$  such that  $a_1p_1 = p + tq_1$  for some  $t \in \mathbb{Z}$ . Notice that  $a_1p_1 < q_1p_1$  and therefore  $t < p_1$ . So  $x = (y - t)\beta_0 + a_1\beta_1$  is the required presentation.

Now assume that  $k \geq 2$  and that a presentation (5.1) exists for  $k - 1$ . Let  $x \in \Gamma_k$  and  $x \geq q_k\beta_k$ , then  $x = \frac{y}{Q_{k-1}} + \frac{p}{Q_{k-1}q_k}$  for some  $p, y \in \mathbb{Z}$  such that  $0 \leq p < q_k$ . Since  $(p_k, q_k) = 1$  there exists an integer  $0 \leq a_k < q_k$  such that  $a_kp_k = p + tq_k$  for some  $t \in \mathbb{Z}$ . Then  $x = \frac{y}{Q_{k-1}} + a_k(\beta_k - q_{k-1}\beta_{k-1}) - \frac{t}{Q_{k-1}}$ , so that  $x - a_k\beta_k = \frac{y-t}{Q_{k-1}} - a_kq_{k-1}\beta_{k-1} \in \Gamma_{k-1}$  and  $x - a_k\beta_k \geq (q_k - a_k)\beta_k \geq \beta_k > q_{k-1}\beta_{k-1}$ . Thus, by the inductive assumption we have that  $x - a_k\beta_k = \sum_{j=0}^{k-1} a_j\beta_j$  with integer coefficients  $0 \leq a_j < q_j$ .

To prove uniqueness of the presentation (5.1) it suffices to show that if  $\sum_{j=0}^k c_j\beta_j = 0$  for some integer coefficients  $-q_j < c_j < q_j$  then  $c_j = 0$  for all  $j$ . We again use induction on  $k$ . The claim is trivial for  $k = 0$ . Assume that the claim is true for

$k-1$  and suppose that  $\sum_{j=0}^k c_j \beta_j = 0$  for some integer coefficients  $-q_j < c_j < q_j$ . We only need to show that  $c_k = 0$ . Since  $c_k \beta_k = -\sum_{j=0}^{k-1} c_j \beta_j$ ,  $c_k \beta_k \in \Gamma_{k-1}$  and therefore  $c_k \frac{p_k}{Q_{k-1} q_k} \in \Gamma_{k-1}$ . Thus  $\frac{c_k p_k}{Q_{k-1} q_k}$  is  $\mathbb{Z}$ -divisible by  $\frac{1}{Q_{k-1}}$ , that is  $q_k | c_k p_k$ . This implies that  $q_k | c_k$ , since  $(p_k, q_k) = 1$ , and therefore  $c_k = 0$  due to the inequality  $-q_k < c_k < q_k$ .  $\square$

**Corollary 5.9.** *For all  $k > 0$  there exists a unique representation  $q_k \beta_k = \sum_{j=0}^{k-1} n_{k,j} \beta_j$  with integer coefficients  $0 \leq n_{k,j} < q_j$ .*

*Proof.* The statement is clear for  $k = 1$ , since  $q_1 \beta_1 = p_1 \beta_0$ . For  $k > 1$  the conclusion follows from Proposition 5.8, since  $q_k \beta_k \in \Gamma_{k-1}$  and  $q_k \beta_k > q_{k-1} \beta_{k-1}$ .  $\square$

**Corollary 5.10.** *In the notations of Section 4, the sequence of jumping polynomials  $\{T_i\}_{i \geq 0}$  in  $S$  is well-defined.*

*Proof.* The statement follows immediately from Corollary 5.9 and the assumption that  $V^*/m_{V^*} = \mathbf{k}$ .  $\square$

We now recall some well-known facts about continued fractions. Let  $p$  and  $q$  be positive integers such that  $(p, q) = 1$ . Consider the Euclidian algorithm for finding the greatest common divisor of  $p$  and  $q$ :

$$\begin{aligned} r_0 &= f_1 r_1 + r_2 \\ r_1 &= f_2 r_2 + r_3 \\ &\dots \\ r_{N-2} &= f_{N-1} r_{N-1} + 1 \\ r_{N-1} &= f_N \cdot 1, \end{aligned}$$

where  $r_0 = p$ ,  $r_1 = q$  and  $r_1 > r_2 > \dots > r_{N-1} > r_N = 1$ . Denote by  $N = N(p, q)$  the number of divisions in the Euclidian algorithm for  $p$  and  $q$  and by  $f_1, f_2, \dots, f_N$  the coefficients in the Euclidian algorithm for  $p$  and  $q$ . Define  $F_i = f_1 + \dots + f_i$  and  $\epsilon(p, q) = f_1 + \dots + f_N = F_N$ ,  $f_1(p, q) = f_1 = \left[ \frac{p}{q} \right]$ . Let  $a$  and  $b$  be integers such that  $0 < a \leq p$ ,  $0 \leq b < q$ , and  $aq - bp = 1$ .

**Remark 5.11.** With notations as above,  $\frac{p}{q} = f_1 + \frac{1}{f_2 + \dots + \frac{1}{f_N}}$ .

Let  $\{P_k(z_1, \dots, z_k)\}_{k \in \mathbb{N}_0}$  be a sequence of polynomials as in [10]. So  $P_k(z_1, \dots, z_k) \in \mathbb{N}_0[z_1, \dots, z_k]$  is a polynomial in  $k$  variables with nonnegative integer coefficients such that for any set of numbers  $c_1, \dots, c_K$  we have

$$c_1 + \frac{1}{c_2 + \dots + \frac{1}{c_K}} = \frac{P_K(c_1, \dots, c_K)}{P_{K-1}(c_2, \dots, c_K)}.$$

We also assume that  $P_0 = 1$  and set  $P_{-1} = 0$ .

Then it follows from properties (1.2)-(1.6) in [10] that

$$\begin{aligned} p &= P_N(f_1, \dots, f_N), \\ q &= P_{N-1}(f_2, \dots, f_N), \\ a &= P_{N-1}(f_1, \dots, f_{N-1}), \quad b = P_{N-2}(f_2, \dots, f_{N-1}), \quad \text{if } N \text{ is odd,} \\ a &= p - P_{N-1}(f_1, \dots, f_{N-1}), \quad b = q - P_{N-2}(f_2, \dots, f_{N-1}), \quad \text{if } N \text{ is even.} \end{aligned}$$

We also recall property (1.5) from [10] here since it will be used in the sequel

$$\begin{aligned} P_k(f_1, \dots, f_k) &= f_k P_{k-1}(f_1, \dots, f_{k-1}) + P_{k-2}(f_1, \dots, f_{k-2}), \\ P_{k-1}(f_2, \dots, f_k) &= f_k P_{k-2}(f_2, \dots, f_{k-1}) + P_{k-3}(f_2, \dots, f_{k-2}). \end{aligned}$$

## 6. SEQUENCES OF QUADRATIC TRANSFORMS

We will now consider a sequence

$$S = S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_i \rightarrow \dots$$

of quadratic transforms along  $\nu^*$ . Suppose that  $E$  is a nonsingular irreducible curve on  $\text{Spec } S$ . Denote by  $\pi_i$  the map  $\text{Spec } S_i \rightarrow \text{Spec } S$  and by  $E_i$  the reduced simple normal crossing divisor  $\pi_i^{-1}(E)_{red}$ . We say that  $S_i$  is *free* if  $E_i$  has exactly one irreducible component. For a free ring  $S_i$  and a regular parameter  $x_i \in S_i$  we will say that  $x_i$  is an *exceptional parameter* if  $x_i$  is supported on  $E_i$ . A system of parameters  $(x_i, y_i)$  of a free ring  $S_i$  is called *permissible* if  $x_i$  is an exceptional parameter.

If  $S_i$  has regular parameters  $(x_i, y_i)$  then we can choose regular parameters  $(x_{i+1}, y_{i+1})$  in  $S_{i+1}$  as follows

- a) if  $\nu^*(x_i) < \nu^*(y_i)$  then  $x_{i+1} = x_i$  and  $y_{i+1} = \frac{y_i}{x_i}$ ,
- b) if  $\nu^*(x_i) > \nu^*(y_i)$  then  $x_{i+1} = \frac{x_i}{y_i}$  and  $y_{i+1} = y_i$ ,
- c) if  $\nu^*(x_i) = \nu^*(y_i)$  then  $x_{i+1} = x_i$  and  $y_{i+1} = \frac{y_i}{x_i} - c$ , where  $c \in \mathbf{k}$  is the residue of  $\frac{y_i}{x_i}$ .

Our goal is to describe explicitly the sequence of quadratic transforms of  $S$  along  $\nu^*$ . Assume that  $(x, y)$  is a permissible system of parameters in  $S$ . Let  $p$  and  $q$  be positive coprime integers such that  $\frac{\nu^*(y)}{\nu^*(x)} = \frac{p}{q}$ . We denote by  $\mu$  the value  $\nu^*(x)$ . Let  $N = N(p, q)$ ,  $f_1, \dots, f_n$  and  $F_1, \dots, F_N$  be defined by the Euclidian algorithm for  $p$  and  $q$  as in Section 5. Let  $a$  and  $b$  be integers such that  $0 < a \leq p$ ,  $0 \leq b < q$  and  $aq - bp = 1$ . We will investigate the following sequence of quadratic transforms along  $\nu^*$

$$S = S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_{F_1} \rightarrow \dots \rightarrow S_{F_j} \rightarrow \dots \rightarrow S_{F_N}.$$

If  $N > 1$  then for all  $0 \leq j \leq F_1$ , the ring  $S_j$  is free and has a permissible system of parameters  $(x, \frac{y}{x^j})$ . In particular,

$$(X_1, Y_1) = \left( x, \frac{y}{x^{f_1}} \right) = \left( \frac{x^{P_0}}{y^{P-1}}, \frac{y^{P_0}}{x^{P_1(f_1)}} \right)$$

is a permissible system of regular parameters in  $S_{F_1}$  with  $\nu^*(X_1) = \mu$  and  $\nu^*(Y_1) = \frac{r_2}{q}\mu$ . If  $N = 1$  then  $S = S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_{F_N}$  is a sequence of free rings and  $S_{F_N}$  has a permissible system of parameters

$$(X_N, Y_N) = (X_1, Y_1) = \left(x, \frac{y}{xf_1} - c\right) = \left(\frac{x^a}{y^b}, \frac{y^q}{x^p} - c\right),$$

where  $c \in \mathbf{k}$  is the residue of  $\frac{y^q}{x^p}$ . Notice also that  $\nu^*(X_N) = \mu = (\nu^*(x), \nu^*(y))$ .

If  $N > 2$  then for all  $0 < j \leq f_2$ , the ring  $S_{F_1+j}$  is not free and has a system of regular parameters  $(\frac{X_1}{Y_1^j}, Y_1)$ . In particular,

$$(X_2, Y_2) = \left(\frac{X_1}{Y_1^{f_2}}, Y_1\right) = \left(\frac{x^{P_0+f_2P_1(f_1)}}{y^{f_2}}, \frac{y^{P_0}}{x^{P_1(f_1)}}\right) = \left(\frac{x^{P_2(f_1, f_2)}}{y^{P_1(f_2)}}, \frac{y^{P_0}}{x^{P_1(f_1)}}\right)$$

are regular parameters in  $S_{F_2}$  and  $\nu^*(X_2) = \frac{r_3}{q}\mu$ ,  $\nu^*(Y_2) = \frac{r_2}{q}\mu$ .

In general, for all  $1 < k < N$  and  $0 < j \leq f_k$ , the ring  $S_{F_{k-1}+j}$  is not free and has a system of regular parameters  $(\frac{X_{k-1}}{Y_{k-1}^j}, Y_{k-1})$  if  $k$  is even or  $(X_{k-1}, \frac{Y_{k-1}}{X_{k-1}^j})$  if  $k$  is odd. In particular, if  $k$  is even then  $S_{F_k}$  has a system of regular parameters

$$(X_k, Y_k) = \left(\frac{X_{k-1}}{Y_{k-1}^{f_k}}, Y_{k-1}\right)$$

where  $\nu^*(X_k) = \frac{r_{k+1}}{q}\mu$  and  $\nu^*(Y_k) = \frac{r_k}{q}\mu$ . We also notice that since

$$\frac{X_{k-1}}{Y_{k-1}^{f_k}} = \frac{x^{P_{k-2}(f_1, \dots, f_{k-2}) + f_k P_{k-1}(f_1, \dots, f_{k-1})}}{y^{P_{k-3}(f_2, \dots, f_{k-2}) + f_k P_{k-2}(f_2, \dots, f_{k-1})}} = \frac{x^{P_k(f_1, \dots, f_k)}}{y^{P_{k-1}(f_2, \dots, f_k)}}$$

the regular parameters  $(X_k, Y_k)$  satisfy the equality

$$(X_k, Y_k) = \left(\frac{x^{P_k(f_1, \dots, f_k)}}{y^{P_{k-1}(f_2, \dots, f_k)}}, \frac{y^{P_{k-2}(f_2, \dots, f_{k-1})}}{x^{P_{k-1}(f_1, \dots, f_{k-1})}}\right).$$

If  $k$  is odd then  $S_{F_k}$  has a system of regular parameters

$$(X_k, Y_k) = \left(X_{k-1}, \frac{Y_{k-1}}{X_{k-1}^{f_k}}\right)$$

where  $\nu^*(X_k) = \frac{r_k}{q}\mu$  and  $\nu^*(Y_k) = \frac{r_{k+1}}{q}\mu$ . We notice that since

$$\frac{Y_{k-1}}{X_{k-1}^{f_k}} = \frac{y^{P_{k-3}(f_2, \dots, f_{k-2}) + f_k P_{k-2}(f_2, \dots, f_{k-1})}}{x^{P_{k-2}(f_1, \dots, f_{k-2}) + f_k P_{k-1}(f_1, \dots, f_{k-1})}} = \frac{y^{P_{k-1}(f_2, \dots, f_k)}}{x^{P_k(f_1, \dots, f_k)}}$$

the regular parameters  $(X_k, Y_k)$  satisfy the equality

$$(X_k, Y_k) = \left(\frac{x^{P_{k-1}(f_1, \dots, f_{k-1})}}{y^{P_{k-2}(f_2, \dots, f_{k-1})}}, \frac{y^{P_{k-1}(f_2, \dots, f_k)}}{x^{P_k(f_1, \dots, f_k)}}\right).$$

Finally, if  $N > 1$  is odd then for all  $0 < j < f_N$  the ring  $S_{F_{N-1}+j}$  is not free and has a system of regular parameters  $(X_{N-1}, \frac{Y_{N-1}}{X_{N-1}^j})$ . Moreover,  $S_{F_N}$  is the first free ring after a sequence of non-free rings  $S_{F_1+1} \rightarrow \dots \rightarrow S_{F_{N-1}}$ . If  $c \in \mathbf{k}$  is the residue of  $\frac{y^q}{x^p}$  then

$$\begin{aligned} (X_N, Y_N) &= \left( X_{N-1}, \frac{Y_{N-1}}{X_{N-1}^{f_N}} - c \right) = \\ &= \left( \frac{x^{P_{N-1}(f_1, \dots, f_{N-1})}}{y^{P_{N-2}(f_2, \dots, f_{N-1})}}, \frac{y^{P_{N-1}(f_2, \dots, f_N)}}{x^{P_N(f_1, \dots, f_N)}} - c \right) = \left( \frac{x^a}{y^b}, \frac{y^q}{x^p} - c \right) \end{aligned}$$

form a permissible system of parameters in  $S_{F_N}$  with  $\nu^*(X_N) = \frac{r_N}{q}\mu = \frac{1}{q}\mu = (\nu^*(x), \nu^*(y))$ .

If  $N > 1$  is even then for all  $0 < j < f_N$  the ring  $S_{F_{N-1}+j}$  is not free and has a system of regular parameters  $(\frac{X_{N-1}}{Y_{N-1}^j}, Y_{N-1})$ . Moreover,  $S_{F_N}$  is the first free ring after a sequence of non-free rings  $S_{F_1+1} \rightarrow \dots \rightarrow S_{F_{N-1}}$ . If  $c \in \mathbf{k}$  is the residue of  $\frac{y^q}{x^p}$  then

$$\begin{aligned} (X_N, Y_N) &= \left( \frac{X_{N-1}}{Y_{N-1}^{f_N}}, \frac{Y_{N-1}}{X_{N-1}} - c \right) = \left( \frac{X_{N-1}}{Y_{N-1}^{f_N}} \cdot Y_{N-1}, \frac{Y_{N-1}}{X_{N-1}} - c \right) = \\ &= \left( \frac{x^{P_N(f_1, \dots, f_N) - P_{N-1}(f_1, \dots, f_{N-1})}}{y^{P_{N-1}(f_2, \dots, f_N) - P_{N-2}(f_2, \dots, f_{N-1})}}, \frac{y^{P_{N-1}(f_2, \dots, f_N)}}{x^{P_N(f_1, \dots, f_N)}} - c \right) = \left( \frac{x^a}{y^b}, \frac{y^q}{x^p} - c \right) \end{aligned}$$

form a permissible system of parameters in  $S_{F_N}$  with  $\nu^*(X_N) = \frac{r_N}{q}\mu = \frac{1}{q}\mu = (\nu^*(x), \nu^*(y))$ .

The following lemma summarizes the above discussion. We will often refer to it in the rest of the paper.

**Lemma 6.1.** *Suppose that  $S$  is a free ring and  $(x, y)$  is a permissible system of parameters in  $S$  such that  $\frac{\nu^*(y)}{\nu^*(x)} = \frac{p}{q}$  for some coprime integers  $p$  and  $q$ . Let  $k = \epsilon(p, q)$ ,  $f_1 = f_1(p, q) = \left\lceil \frac{p}{q} \right\rceil$  and let  $a$  and  $b$  be nonnegative integers such that  $a \leq p$ ,  $b < q$ , and  $aq - bp = 1$ . Then the sequence of quadratic transforms along  $\nu^*$*

$$S = S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_{f_1} \rightarrow S_{f_1+1} \rightarrow \dots \rightarrow S_{k-1} \rightarrow S_k \quad (6.1)$$

has the following properties:

- 1)  $S_0, S_1, \dots, S_{f_1}$  and  $S_k$  are free rings.
- 2) Non-free rings appear in (6.1) if and only if  $k > f_1$ , that is if  $q > 1$ . In this case  $S_{f_1+1}, \dots, S_{k-1}$  are non-free.

3)  $S_k$  has a permissible system of coordinates  $(X, Y) = \left( \frac{x^a}{y^b}, \frac{y^q}{x^p} - c \right)$ , where  $c \in \mathbf{k}$  is the residue of  $\frac{y^q}{x^p}$ . Moreover,  $\nu^*(X) = (\nu^*(x), \nu^*(y)) = \frac{1}{q}\nu^*(x)$  and  $x = X^q(Y + c)^b$ ,  $y = X^p(Y + c)^a$ .

*Proof.* We only check that  $X^q(Y + c)^b = \frac{x^{aq}}{y^{bq}} \cdot \frac{y^{qb}}{x^{pb}} = x^{aq-bp} = x$  and  $X^p(Y + c)^a = \frac{x^{ap}}{y^{bp}} \cdot \frac{y^{qa}}{x^{pa}} = y^{aq-bp} = y$ .  $\square$

## 7. PROPERTIES OF JUMPING POLYNOMIALS

In this section assumptions and notations are as in Section 4. We fix regular parameters  $(x, y)$  of  $S$  and we further assume that  $(x, y)$  is a permissible system of parameters in  $S$  by setting  $E$  to be the curve on  $\text{Spec } S$  defined by  $x = 0$ .

For all  $k > 0$  let  $d_k = (p_1, p_2, \dots, p_k)$ . We will use this notation often in the rest of the paper.

**Theorem 7.1.** *Suppose that  $R$  is a regular local ring dominated by  $S$  and  $(u, v)$  are regular parameters of  $R$  such that*

$$\begin{aligned} u &= x^t \\ v &= y, \end{aligned}$$

where  $t$  is a positive integer.

If  $t|d_k$  for some  $k > 0$  then  $\{u, \{T_i\}_{i=1}^{k+1}\} = \{T'_i\}_{i=0}^{k+1}$  is the beginning of a sequence of jumping polynomials in  $R$ . Moreover, for all  $1 \leq i \leq k$  the pair of coprime integers defined in the construction of jumping polynomials  $\{T'_i\}_{i \geq 0}$  in  $R$  is  $(p'_i, q'_i) = \left( \frac{p_i}{t}, q_i \right)$ .

*Proof.* Since  $\nu^*(u) = t$  in order to construct the sequence of jumping polynomials  $\{T'_i\}_{i \geq 0}$  in  $R$  we use the following valuation  $\tilde{\nu}$  of  $K^*$ :

$$\tilde{\nu}(f) = \frac{\nu^*(f)}{t} \text{ for all } f \in K^*.$$

We have  $T'_1 = v = y = T_1$  and the coprime integers  $p'_1$  and  $q'_1$  are such that  $\frac{p'_1}{q'_1} = \tilde{\nu}(y) = \frac{p_1}{tq_1}$ . Assume  $t|p_1$ . Since  $(p_1, q_1) = 1$  we get  $p'_1 = \frac{p_1}{t}$  and  $q'_1 = q_1$ . Then  $T'_2 = v^{q_1} - \lambda'_1 u^{p'_1} = y^{q_1} - \lambda'_1 x^{p_1}$ , where  $\lambda'_1$  is the residue of  $\frac{v^{q_1}}{u^{p'_1}} = \frac{y^{q_1}}{x^{p_1}}$ , that is  $\lambda'_1 = \lambda_1$  and  $T'_2 = T_2$ . The statement is proved for  $k = 1$ .

By induction on  $k$  it suffices to show that the statement holds for  $k$  provided it holds for  $k - 1$ . Then since  $t|d_k$  and  $d_k|d_{k-1}$  by the inductive assumption we have

$T'_i = T_i$  for all  $1 \leq i \leq k$  and  $(p'_i, q'_i) = (\frac{p_i}{t}, q_i)$  for all  $1 \leq i \leq k-1$ . The coprime integers  $p'_k$  and  $q'_k$  satisfy the following equality

$$\frac{p'_k}{q'_k} = Q_{k-1}(\tilde{\nu}(T_k) - q_{k-1}\tilde{\nu}(T_{k-1})) = \frac{1}{t}Q_{k-1}(\nu^*(T_k) - q_{k-1}\nu^*(T_{k-1})) = \frac{1}{t} \cdot \frac{p_k}{q_k}.$$

Since  $t|p_k$  and  $(p_k, q_k) = 1$  we get  $p'_k = \frac{p_k}{t}$  and  $q'_k = q_k$ . Then

$$T'_{k+1} = (T'_k)^{q'_k} - \lambda'_k \prod_{i=0}^{k-1} (T'_i)^{n'_{k,i}} = T_k^{q_k} - \lambda'_k u^{n'_{k,0}} \prod_{i=1}^{k-1} T_i^{n'_{k,i}},$$

where  $\lambda'_k$  is the residue of  $\frac{T_k^{q_k}}{u^{n'_{k,0}} \prod_{i=1}^{k-1} T_i^{n'_{k,i}}}$  and  $q_k \tilde{\nu}(T_k) = n'_{k,0} + \sum_{i=1}^{k-1} n'_{k,i} \tilde{\nu}(T_i)$  with  $n'_{k,i} < q_i$  for all  $1 \leq i \leq k-1$ . We notice that the last equality is equivalent to

$$q_k \nu^*(T_k) = n'_{k,0} t + \sum_{i=1}^{k-1} n'_{k,i} \nu^*(T_i).$$

We also have

$$q_k \nu^*(T_k) = n_{k,0} + \sum_{i=1}^{k-1} n_{k,i} \nu^*(T_i) \quad (7.1)$$

from the construction of jumping polynomials in  $S$ . Thus from the uniqueness of presentation (7.1) we obtain  $n'_{k,i} = n_{k,i}$  for all  $1 \leq i \leq k-1$  and  $n'_{k,0} = \frac{n_{k,0}}{t}$ . Thus,  $u^{n'_{k,0}} \prod_{i=1}^{k-1} T_i^{n'_{k,i}} = \prod_{i=0}^{k-1} T_i^{n_{k,i}}$ ,  $\lambda'_k = \lambda_k$ , the residue of  $\frac{T_k^{q_k}}{\prod_{i=0}^{k-1} T_i^{n_{k,i}}}$ , and  $T'_{k+1} = T_{k+1}$ . This completes the proof.  $\square$

**Corollary 7.2.** *For all  $k > 0$  all powers of  $x$  that appear in  $T_2, T_3, \dots, T_{k+1}$  are multiples of  $d_k$ .*

*Proof.* Notice that Theorem 7.1 in particular shows that if  $t|d_k$ , then all powers of  $x$  that appear in  $T_2, T_3, \dots, T_{k+1}$  are multiples of  $t$ .  $\square$

Our next goal is to describe the images of jumping polynomials under blowups of  $S$  along  $\nu^*$ .

Before we state the next results we notice that if  $S \subset \bar{S}$  is any subring of the  $m_S$ -adic completion  $\hat{S}$  of  $S$  we can extend the valuation  $\nu^*$  to a valuation of  $\bar{K} = QF(\bar{S})$  centered in  $\bar{S}$ . We first consider the unique extension  $\hat{\nu}$  of  $\nu^*$  to  $\hat{K} = QF(\hat{S})$  centered in  $\hat{S}$ , then we restrict  $\hat{\nu}$  to  $\bar{K}$ . By abuse of notations we will say that  $\nu^*$  is also a valuation of  $\bar{K}$ .

We will be mostly interested in the case where  $\bar{S}$  is an étale extension of  $S$ . If  $n$  is a maximal ideal of  $\bar{S}$ , we will say that the map  $S \rightarrow \bar{S}_n$  is *local étale*. Most of the times we will have  $n = m_{\nu^*} \cap \bar{S}$ , the center of the valuation.

We will first consider a sequence of ring extensions

$$S = S'_0 \rightarrow \bar{S}'_0 \rightarrow S'_1 \rightarrow \dots \rightarrow \bar{S}'_{i-1} \rightarrow S'_i \rightarrow \dots$$

such that for all  $i \geq 0$   $\bar{S}'_i \rightarrow S'_{i+1}$  is a quadratic transform along  $\nu^*$  and  $S'_i \rightarrow \bar{S}'_i$  is a local étale extension. As before let  $E$  be a nonsingular irreducible curve on  $\text{Spec } S$ , denote by  $\pi'_i$  the map  $\text{Spec } S'_i \rightarrow \text{Spec } S$  and by  $E'_i$  the reduced simple normal crossing divisor  $\pi'^{-1}_i(E)_{red}$ .

In what follows, for all  $i > 0$  let  $a_i, b_i$  be nonnegative integers such that  $a_i q_i - b_i p_i = 1$  and  $a_i \leq p_i, b_i < q_i$ . The existence of  $a_i$  and  $b_i$  is due to the Euclidean division algorithm. Let  $k_0 = 0$  and  $k_i = k_{i-1} + \epsilon(p_i, q_i)$  ( $\epsilon(p_i, q_i)$  was defined in Section 5).

**Lemma 7.3.** *There exists a sequence of ring extensions*

$$\begin{aligned} S = S'_0 \rightarrow S'_1 \rightarrow \dots \rightarrow S'_{k_1} \rightarrow \bar{S}'_{k_1} \rightarrow S'_{k_1+1} \rightarrow \dots \\ \dots \rightarrow S'_{k_2} \rightarrow \bar{S}'_{k_2} \rightarrow \dots \rightarrow S'_{k_i} \rightarrow \bar{S}'_{k_i} \rightarrow \dots \end{aligned}$$

such that for all  $i > 0$  and  $j \neq k_i, S'_j \rightarrow S'_{j+1}$  and  $\bar{S}'_{k_i} \rightarrow S'_{k_i+1}$  are quadratic transforms along  $\nu^*$ ,  $\bar{S}'_{k_i} = S'_{k_i}[\alpha_i]_{m_{V^*} \cap S'_{k_i}[\alpha_i]}$  are local étale extensions and the following hold:

- 1)  $\alpha_i^{p_i+1} \in S'_{k_i}$  is a unit.
- 2)  $S'_{k_i}$  is free and has a permissible system of parameters  $(z_i, w_i)$  such that  $z_i$  is an exceptional parameter and  $w_i = \frac{T_{i+1}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}}$  is the strict transform of  $T_{i+1}$  in  $S'_{k_i}$ .
- 3)  $\nu^*(z_i) = \frac{1}{Q_i}$  and  $\nu^*(w_i) = \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}$ .
- 4) For all  $0 \leq j \leq i$ ,  $T_j = z_i^{Q_i \beta_i} \tau_{j,i}$ , where  $\tau_{j,i} \in S'_{k_i}$  is a unit.

*Proof.* We apply induction on  $i$ . For  $i = 1$ , by Lemma 6.1 the ring  $S'_{k_1}$  is free and has a system of regular parameters  $(z_1, w_1)$ , where

$$\begin{aligned} z_1 &= \frac{x^{a_1}}{y^{b_1}} & \nu^*(z_1) &= \left(1, \frac{p_1}{q_1}\right) = \frac{1}{q_1} \\ w_1 &= \frac{y^{q_1} - \lambda_1 x^{p_1}}{x^{p_1}} = \frac{T_2}{T_0^{n_{1,0}}} & \nu^*(w_1) &= \beta_2 - p_1 = \frac{1}{q_1} \cdot \frac{p_2}{q_2}. \end{aligned}$$

We also have  $T_0 = x = z_1^{q_1} (w_1 + \lambda_1)^{b_1} = z_1^{q_1} \tau_{0,1}$  and  $T_1 = y = z_1^{p_1} (w_1 + \lambda_1)^{a_1} = z_1^{p_1} \tau_{1,1}$ , where  $\tau_{0,1}$  and  $\tau_{1,1}$  are units in  $S'_{k_1}$ .

Now assume that the lemma is true for  $i - 1$ . We set  $\alpha_{i-1} = \left( \frac{\prod_{j=0}^{i-1} \tau_{j,i-1}^{n_{i,j}}}{\prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j} q_i}} \right)^{\frac{1}{p_i}}$  and  $\bar{S}'_{k_{i-1}} = S'_{k_{i-1}}[\alpha_{i-1}]_{m_{V^*} \cap S'_{k_{i-1}}[\alpha_{i-1}]}$ . Then  $\alpha_{i-1}^{p_i} \in S'_{k_{i-1}}$  and  $\bar{S}'_{k_{i-1}}$  is a local étale extension of  $S'_{k_{i-1}}$ . Let  $\bar{z}_{i-1} = z_{i-1} \alpha_{i-1}$ , then  $(\bar{z}_{i-1}, w_{i-1})$  is a permissible system of parameters in  $\bar{S}'_{k_{i-1}}$ ,  $\nu^*(\bar{z}_{i-1}) = \nu(z_{i-1}) = \frac{1}{Q_{i-1}}$  and  $\nu^*(w_{i-1}) = \frac{1}{Q_{i-1}} \cdot \frac{p_i}{q_i}$ . Recall

that  $k_i = k_{i-1} + \epsilon(p_i, q_i)$ . Therefore, by Lemma 6.1 the ring  $S'_{k_i}$  is free and has a permissible system of parameters  $(z_i, w_i)$  such that

$$z_i = \frac{\bar{z}_{i-1}^{a_i}}{w_{i-1}^{b_i}} \quad \text{is an exceptional parameter with } \nu^*(z_i) = \left( \frac{1}{Q_{i-1}}, \frac{1}{Q_{i-1}} \cdot \frac{p_i}{q_i} \right) = \frac{1}{Q_i},$$

$$w_i = \frac{w_{i-1}^{q_i}}{\bar{z}_{i-1}^{p_i}} - c_i, \quad \text{where } c_i \in \mathbf{k} \text{ is the residue of } \frac{w_{i-1}^{q_i}}{\bar{z}_{i-1}^{p_i}}.$$

We notice that for all  $0 \leq l \leq i-1$  the following equalities hold

$$\begin{aligned} \prod_{j=0}^l T_j^{n_{l+1,j}} &= \prod_{j=0}^l z_{i-1}^{Q_{i-1}\beta_j n_{l+1,j}} \tau_{j,i-1}^{n_{l+1,j}} = z_{i-1}^{Q_{i-1} \sum_{j=0}^l n_{l+1,j} \beta_j} \prod_{j=0}^l \tau_{j,i-1}^{n_{l+1,j}} \\ &= z_{i-1}^{Q_{i-1} q_{l+1} \beta_{l+1}} \prod_{j=0}^l \tau_{j,i-1}^{n_{l+1,j}} = z_{i-1}^{Q_{i-1} q_{l+1} \beta_{l+1}} \alpha_{i-1}^{-Q_{i-1} q_{l+1} \beta_{l+1}} \prod_{j=0}^l \tau_{j,i-1}^{n_{l+1,j}}. \end{aligned}$$

Hence,

$$\prod_{j=0}^{i-1} T_j^{n_{i,j}} = \bar{z}_{i-1}^{Q_i \beta_i} \alpha_{i-1}^{-Q_i \beta_i} \prod_{j=0}^{i-1} \tau_{j,i-1}^{n_{i,j}}, \quad \prod_{j=0}^{i-2} T_j^{n_{i-1,j}} = \bar{z}_{i-1}^{Q_{i-1} q_{i-1} \beta_{i-1}} \alpha_{i-1}^{-Q_{i-1} q_{i-1} \beta_{i-1}} \prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j}}$$

and

$$\begin{aligned} \frac{w_{i-1}^{q_i}}{\bar{z}_{i-1}^{p_i}} &= \frac{T_i^{q_i}}{\bar{z}_{i-1}^{p_i} \prod_{j=0}^{i-2} T_j^{n_{i-1,j} q_i}} = \frac{T_i^{q_i}}{\bar{z}_{i-1}^{Q_i q_{i-1} \beta_{i-1} + p_i} \alpha_{i-1}^{-Q_i q_{i-1} \beta_{i-1}} \prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j} q_i}} \\ &= \frac{T_i^{q_i}}{\bar{z}_{i-1}^{Q_i \beta_i} \alpha_{i-1}^{-Q_i \beta_i + p_i} \prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j} q_i}} = \frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} \cdot \frac{\prod_{j=0}^{i-1} \tau_{j,i-1}^{n_{i,j}}}{\prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j} q_i}} \alpha_{i-1}^{-p_i} \\ &= \frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}}. \end{aligned} \tag{7.2}$$

Therefore,  $c_i$  coincides with  $\lambda_i$ , and we get  $w_i = \frac{T_{i+1}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}}$  with  $\nu^*(w_i) = \beta_{i+1} - q_i \beta_i =$

$$\frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

Finally, in view of Lemma 6.1 we have  $z_{i-1} = z_i^{q_i} (w_i + \lambda_i)^{b_i} \alpha_{i-1}^{-1}$  and  $w_{i-1} = z_i^{p_i} (w_i + \lambda_i)^{a_i}$ . Notice also that  $Q_{i-1} \beta_j$  is an integer for all  $0 \leq j \leq i-1$ . Thus, for all  $0 \leq j \leq i-1$  we get

$$T_j = z_{i-1}^{Q_{i-1} \beta_j} \tau_{j,i-1} = z_i^{Q_i \beta_j} ((w_i + \lambda_i)^{b_i} \alpha_{i-1}^{-1})^{Q_{i-1} \beta_j} \tau_{j,i-1} = z_i^{Q_i \beta_j} \tau_{j,i},$$

where  $\tau_{j,i}$  is a unit in  $S'_{k_i}$ , and

$$\begin{aligned} T_i &= w_{i-1} \prod_{j=0}^{i-2} T_j^{n_{i-1,j}} = w_{i-1} z_{i-1}^{Q_{i-1}q_{i-1}\beta_{i-1}} \prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j}} = \\ &= z_i^{Q_i q_{i-1} \beta_{i-1} + p_i} (w_i + \lambda_i)^{a_i} ((w_i + \lambda_i)^{b_i} \alpha_{i-1}^{-1})^{Q_{i-1} q_{i-1} \beta_{i-1}} \prod_{j=0}^{i-2} \tau_{j,i-1}^{n_{i-1,j}} = z_i^{Q_i \beta_i} \tau_{i,i}, \end{aligned}$$

where  $\tau_{i,i}$  is a unit in  $S'_{k_i}$ . This completes the proof of the lemma.  $\square$

**Remark 7.4.** In our set-up, assume that  $\bar{S}$  is a local étale extension of  $S$ ,  $S'$  is a quadratic transform of  $S$  along  $\nu^*$ , and  $\bar{S}'$  is a quadratic transform of  $\bar{S}$  along  $\nu^*$ . Without loss of generality, assume that  $S$  has regular parameters  $(x, y)$  with  $\nu^*(y) \geq \nu^*(x)$ . Then  $S' = S[x, \frac{y}{x}]_{(x, \frac{y}{x} - \beta)}$ , for some  $\beta \in k$ . Since  $m_{\bar{S}} \bar{S} = m_{\bar{S}'}$ , we have that  $\bar{S}$  has regular parameters  $(x, y)$  and so  $\bar{S}' = S'[x, \frac{y}{x}]_{(x, \frac{y}{x} - \beta')}$ , for some  $\beta' \in k$ . Since the quadratic transforms are along  $\nu^*$ , it follows that  $\beta = \beta'$ . Since  $\bar{S}'$  is essentially of finite type over  $S'$  and  $m_{\bar{S}'} \bar{S}' = m_{\bar{S}'}$ , we have that  $\bar{S}'$  is a local étale extension of  $S'$ . Furthermore, if  $\bar{S} = S[\alpha^{1/n}]_{m_{V^*} \cap S[\alpha^{1/n}]}$ , where  $\alpha \in S$ , we have that  $\bar{S}' = S'[\alpha^{1/n}]_{m_{V^*} \cap S'[\alpha^{1/n}]}$ .

**Theorem 7.5.** *There exists a sequence of quadratic transforms along  $\nu^*$*

$$S = S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_{k_1-1} \rightarrow S_{k_1} \rightarrow S_{k_1+1} \rightarrow \dots \rightarrow S_{k_i} \rightarrow \dots$$

such that for all  $i > 0$ ,  $S_{k_i}$  is free and has a system of regular parameters  $(x_i, y_i)$  such that  $x_i$  is an exceptional parameter,  $T_j = x_i^{Q_i \beta_j} \gamma_{j,i}$  for  $0 \leq j \leq i$ , where  $\gamma_{j,i} \in S_{k_i}$  is a unit,  $\nu^*(x_i) = \frac{1}{Q_i}$ ,  $y_i = \frac{T_{i+1}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}}$  is the strict transform of  $T_{i+1}$  in  $S_{k_i}$  and

$$\nu^*(y_i) = \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

*Proof.* We shall construct the required sequence from the sequence

$$S = S'_0 \rightarrow S'_1 \rightarrow \dots \rightarrow S'_{k_1} \rightarrow \bar{S}'_{k_1} \rightarrow S'_{k_1+1} \rightarrow \dots \rightarrow S'_{k_2} \rightarrow \bar{S}'_{k_2} \rightarrow \dots \rightarrow S'_{k_i} \rightarrow \bar{S}'_{k_i} \rightarrow \dots$$

of Lemma 7.3. It suffices to show, by induction on  $i$ , that we can construct a sequence

$$S = S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_{k_1} \rightarrow S_{k_1+1} \rightarrow \dots \rightarrow S_{k_i} \quad (7.3)$$

with the required properties and such that  $\bar{S}'_{k_i} = S_{k_i}[\alpha_1, \dots, \alpha_i]_{m_{V^*} \cap S_{k_i}[\alpha_1, \dots, \alpha_i]}$  is a local étale extension of  $S_{k_i}$ , where  $\alpha_j^{p_j+1} \in S_{k_i}$  is a unit for  $j = 1, \dots, i$ .

For  $i = 1$ , the sequence (7.3) is given by taking  $S_j = S'_j$  for any  $0 \leq j \leq k_1$  and setting  $(x_1, y_1) = (z_1, w_1)$ . We also notice that  $\bar{S}'_{k_1} = S_{k_1}[\alpha_1]_{m_{V^*} \cap S_{k_1}[\alpha_1]}$ , where  $\alpha_1^{p_1+1} \in S_{k_1}$  is a unit. In general, suppose that the sequence (7.3) has been constructed for  $i - 1$ , i.e., we have a sequence of ring extensions

$$S = S_0 \rightarrow \dots \rightarrow S_{k_1} \rightarrow \dots \rightarrow S_{k_{i-1}} \rightarrow \bar{S}'_{k_{i-1}} \rightarrow S'_{k_{i-1}+1} \rightarrow \dots \rightarrow S'_{k_i} \rightarrow \bar{S}'_{k_i} \rightarrow \dots \quad (7.4)$$

where  $S_{j-1} \rightarrow S_j$  is a quadratic transform for  $1 \leq j \leq k_{i-1}$ ,

$$\bar{S}'_{k_{i-1}} = S_{k_{i-1}}[\alpha_1, \dots, \alpha_{i-1}]_{m_{V^*} \cap S_{k_{i-1}}[\alpha_1, \dots, \alpha_{i-1}]}$$

is a local étale extension (here,  $\alpha_j^{p_j+1} \in S_{k_{i-1}}$  is a unit for  $j = 1, \dots, i-1$ ) and  $\bar{S}'_{k_{i-1}} \rightarrow S'_{k_{i-1}+1}$ ,  $S'_{j-1} \rightarrow S'_j$  is a quadratic transform for  $k_{i-1} + 2 \leq j \leq k_i$ . By applying Remark 7.4 to the subsequence

$$\bar{S}'_{k_{i-1}} \rightarrow S'_{k_{i-1}+1} \rightarrow \dots \rightarrow S'_{k_i}$$

of the sequence (7.4) we obtain a new sequence of ring extensions

$$S = S_0 \rightarrow \dots \rightarrow S_{k_{i-1}} \rightarrow S_{k_{i-1}+1} \rightarrow \dots \rightarrow S_{k_i} \rightarrow \bar{S}'_{k_i} \rightarrow \dots$$

where  $S_{j-1} \rightarrow S_j$  is a quadratic transform for all  $1 \leq j \leq k_i$  and

$$\bar{S}'_{k_i} = S_{k_i}[\alpha_1, \dots, \alpha_i]_{m_{V^*} \cap S_{k_i}[\alpha_1, \dots, \alpha_i]}$$

is a local étale extension (here,  $\alpha_j^{p_j+1} \in S_{k_i}$  is a unit for  $j = 1, \dots, i$ ).

Let  $(x_{i-1}, y_{i-1})$  be a system of parameters in  $S_{k_{i-1}}$  satisfying the required properties. Then by Lemma 6.1 we get that  $x_i = \frac{x_{i-1}^{a_i}}{y_{i-1}^{b_i}}$  is an exceptional parameter of  $S_{k_i}$  with  $\nu^*(x_i) = \frac{1}{Q_{i-1}}$  and  $\delta = \frac{y_{i-1}^{q_i}}{x_{i-1}^{p_i}}$  is a unit in  $S_{k_i}$ . Moreover, by computations similar to (7.2), we have

$$\frac{y_{i-1}^{q_i}}{x_{i-1}^{p_i}} = \frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} \cdot \frac{\prod_{j=0}^{i-1} \gamma_{j,i-1}^{n_{i,j}}}{\prod_{j=0}^{i-2} \gamma_{j,i-1}^{n_{i-1,j} q_i}} = \frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} \theta,$$

where  $\theta \in S_{k_i}$  is a unit.

This implies that  $\frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} \in S_{k_i}$  and, therefore,  $\frac{T_i^{q_i}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} - \lambda_i = \frac{T_{i+1}}{\prod_{j=0}^{i-1} T_j^{n_{i,j}}} = w_i \in S_{k_i}$ . Since  $(z_i, w_i)$  form a permissible system of parameters in  $\bar{S}'_{k_i}$ , by replacing the exceptional parameter  $z_i$  by  $x_i$  we get a permissible system of parameters  $(x_i, w_i)$  in  $S_{k_i}$ . So we will choose a system of regular parameters  $(x_i, y_i)$  in  $S_{k_i}$  by letting  $(x_i, y_i) = \left( \frac{x_{i-1}^{a_i}}{y_{i-1}^{b_i}}, w_i \right)$ . Notice that  $\nu^*(y_i) = \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}$ .

We further have  $x_{i-1} = x_i^{q_i} \delta^{b_i}$  and  $y_{i-1} = x_i^{p_i} \delta^{a_i}$ . Thus, recalling that  $Q_{i-1} \beta_j$  is an integer for all  $0 \leq j \leq i-1$ , we obtain

$$T_j = x_{i-1}^{Q_{i-1} \beta_j} \gamma_{j,i-1} = x_i^{Q_i \beta_j} \delta^{b_i Q_{i-1} \beta_j} \gamma_{j,i-1} = x_i^{Q_i \beta_j} \gamma_{j,i},$$

where  $\gamma_{j,i} \in S_{k_i}$  is a unit, and for  $j = i$  we have

$$\begin{aligned} T_i &= y_{i-1} \prod_{j=0}^{i-2} T_j^{n_{i-1,j}} = x_i^{p_i} \delta^{a_i} \left( \prod_{j=0}^{i-2} x_i^{n_{i-1,j} Q_i \beta_j} \gamma_{j,i}^{n_{i-1,j}} \right) = \\ &= x_i^{p_i + Q_i \sum_{j=0}^{i-2} n_{i-1,j} \beta_j} \left( \delta^{a_i} \prod_{j=0}^{i-2} \gamma_{j,i}^{n_{i-1,j}} \right) = x_i^{p_i + Q_i q_{i-1} \beta_{i-1}} \gamma_{i,i} = x_i^{Q_i \beta_i} \gamma_{i,i} \end{aligned}$$

where  $\gamma_{i,i} \in S_{k_i}$  is a unit.

Hence, the sequence of ring extensions  $S = S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_{k_i} \rightarrow \bar{S}'_{k_i}$  has required properties. The result is proved.  $\square$

**7.1. Remarks on generating sequences.** Assume that the value group of  $\nu^*$  is a non-discrete subgroup of  $\mathbb{Q}$ . Suppose that

$$S = S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_j \rightarrow \dots \quad (7.5)$$

is a sequence of quadratic transforms along  $\nu^*$  and  $E_j$  is the exceptional divisor on  $S_j$  for all  $j \geq 0$ . Then a generating sequence of  $\nu^*$  can be defined as in [5].

**Definition 7.6.** Set  $s'_1 = \bar{s}_0 = 0$ . For all  $i > 0$  let  $(s'_{i+1}, \bar{s}_i)$  be the pair of integers with the following properties:

- 1)  $\bar{s}_i$  is the biggest integer  $s \geq s'_i$  such that  $S_{s'}$  is free for all  $s'$  with  $s'_i \leq s' \leq s$ ;
- 2)  $s'_{i+1}$  is the smallest integer  $s > \bar{s}_i$  such that  $S_s$  is free.

We notice here that the set of free  $S_j$  in (7.5) is infinite, as it follows from Theorem 7.5. Thus the sequences of integers  $\{s'_i\}_{i>0}$  and  $\{\bar{s}_i\}_{i\geq 0}$  are well defined.

**Definition 7.7.** Let  $\{Q_i\}_{i\geq 0}$  be a sequence of elements in  $S$  such that  $Q_0$  is an exceptional parameter in  $S$ ,  $(Q_0, Q_1)$  form a system of parameters in  $S$  and the strict transform of  $\text{div}(Q_1)$  in  $\text{Spec } S_{\bar{s}_1}$  is not empty. For each  $i \geq 2$  let  $\text{div}(Q_i)$  be an analytically irreducible curve in  $\text{Spec } S$  such that the strict transform of  $\text{div}(Q_i)$  in  $\text{Spec } S_{\bar{s}_i}$  is smooth and transversal to  $E_{\bar{s}_i}$ .  $\{Q_i\}_{i\geq 0}$  is a generating sequence of  $\nu^*$  [10].

We show that the set of all independent jumping polynomials in  $S$  satisfies Definition 7.7. Therefore we have an alternative argument that such set forms a generating sequence of  $\nu^*$ . We will need to use the irreducibility criterion of Cossart and Moreno-Socías [2, Theorem 6.2] in the form of Remark 7.17 of [5]. Before we state this irreducibility criterion we recall the notations of Section 4.

Suppose that  $(x, y)$  are permissible parameters in  $S$  and the value group  $\Gamma^*$  of  $\nu^*$  is normalized so that  $\nu^*(x) = 1$ . Let  $\{T_i\}_{i\geq 0}$  denote the sequence of jumping polynomials in  $S$ ,  $\{H_l\}_{l\geq 0}$  denote the sequence of independent jumping polynomials in  $S$  and  $\{i_l\}_{l\geq 0}$  denote the sequence of indexes such that  $H_l = T_{i_l}$ . Then for all  $l \geq 1$  we have

$$H_{l+1} = H_l^{\bar{q}_l} - \lambda_{i_l} \prod_{j=0}^{l-1} H_j^{n_{i_l, i_j}} - \sum_{i'=i_l+1}^{i_{l+1}-1} \lambda_{i'} \prod_{j=0}^l H_j^{n_{i', i_j}}, \quad (7.6)$$

where  $0 \leq n_{i_l, i_j}, n_{i', i_j} < \bar{q}_j$  for all  $i_l < i' < i_{l+1}$  and all  $0 \leq j \leq l$ .

**Theorem 7.8.** (*Remark 7.17, [5]*)

Given a sequence of Weierstrass polynomials  $\{H_l\}_{l \geq 1}$  satisfying (7.6) for all  $l \geq 1$ , set  $\bar{\gamma}_0 = 1$  and define by induction on  $l$  the values

$$\bar{\gamma}_l = \frac{1}{\bar{q}_l} \sum_{j=0}^{l-1} n_{i_l, i_j} \bar{\gamma}_j.$$

Let  $\Gamma_l = \langle \bar{\gamma}_0, \bar{\gamma}_1, \dots, \bar{\gamma}_l \rangle$ . Then  $\{H_l\}_{l \geq 0}$  is a generating sequence of a (uniquely determined) valuation ring  $\bar{V}$  of  $\hat{S} = \mathbf{k}[[x, y]]$ , whose value group is a non-discrete subgroup of  $\mathbb{Q}$  if for  $l > 0$  the  $\bar{\gamma}_l$ 's satisfy the following three properties:

- 1)  $\bar{q}_l \bar{\gamma}_l$  has order precisely  $\bar{q}_l$  in  $\frac{\Gamma_{l-1}}{\bar{q}_l \Gamma_{l-1}}$ ,
- 2)  $\bar{\gamma}_{l+1} > \bar{q}_l \bar{\gamma}_l$ ,
- 3)  $\sum_{j=0}^l n_{i', i_j} \bar{\gamma}_j > \bar{q}_l \bar{\gamma}_l$  for all  $i_l < i' < i_{l+1}$ .

It follows from the construction of jumping polynomials that the values  $\bar{\gamma}_l$ , defined for the sequence of independent jumping polynomials, coincide with  $\bar{\beta}_l$  for all  $l \geq 0$ . Combining Remark 4.3 and Corollary 5.6 we see that independent jumping polynomials satisfy the conditions of Theorem 7.8. Thus the sequence of independent jumping polynomials  $\{H_l\}_{l \geq 0}$  is a generating sequence for some valuation  $\bar{\nu}$  of  $S$ . In particular, every element  $H_l \in S$  is analytically irreducible in  $S$ .

Let the sequence of quadratic transform of  $S$

$$S = S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_{k_1} \rightarrow \dots S_{k_2} \rightarrow \dots \rightarrow S_{k_i} \rightarrow \dots$$

be as in Theorem 7.5. Suppose that  $(x_{i-1}, y_{i-1})$  are permissible regular parameters of  $S_{k_{i-1}}$ , such that  $\nu^*(x_{i-1}) = \frac{1}{Q_{i-1}}$  and  $\nu^*(y_{i-1}) = \frac{1}{Q_{i-1}} \cdot \frac{p_i}{q_i}$ . Lemma 6.1 shows that non-free rings will appear in the subsequence  $S_{k_{i-1}} \rightarrow \dots \rightarrow S_{k_i}$  if and only if  $q_i > 1$ , that is if  $\beta_i$  is an independent  $j$ -value and  $i = i_l$  for some  $l > 0$ . In this case the last free ring in this subsequence is  $S_{k_{i-1} + \left[ \frac{p_i}{q_i} \right]}$  and  $S_{k_i}$  is the first free ring following  $S_{k_{i-1} + \left[ \frac{p_i}{q_i} \right]}$ . Thus for all  $l > 0$ ,  $\bar{s}_l = k_{i_{l-1}} + \left[ \frac{p_{i_l}}{q_{i_l}} \right]$  and  $s'_{l+1} = k_{i_l}$ .

Since  $H_0 = x$  and  $H_1 = y - \sum_{j=1}^{i_1-1} \lambda_j x^{\beta_j}$ ,  $(H_0, H_1)$  form a permissible system of parameters in  $S$ .

By Theorem 7.5 there exists a permissible system of parameters  $(x_{i_{l-1}}, y_{i_{l-1}})$  in  $S_{k_{i_{l-1}}}$  such that  $y_{i_{l-1}}$  is the strict transform of  $H_l$  in  $S_{k_{i_{l-1}}}$ . If  $f_1 = \left[ \frac{p_{i_l}}{q_{i_l}} \right]$  then

$S_{k_{i-1}+f_1} = S_{\bar{s}_i}$  and

$$(X_l, Y_l) = (x_{i-1}, \frac{y_{i-1}}{x_{i-1}^{f_1}})$$

form a permissible system of parameters in  $S_{\bar{s}_i}$ . Thus the strict transform of  $H_l$  in  $S_{\bar{s}_i}$  is  $Y_l$ . In particular, the strict transform of  $\text{div}(H_1)$  in  $\text{Spec } S_{\bar{s}_1}$  is not empty, and for  $l \geq 2$ , the strict transform of  $\text{div}(H_l)$  in  $\text{Spec } S_{\bar{s}_i}$  is smooth and transversal to  $E_{\bar{s}_i}$ . Therefore  $\{H_l\}_{l \geq 0} \subset S$  form a generating sequence of  $\nu^*$ .

## 8. MONOMIALIZATION OF GENERATING SEQUENCES

The goal of this section is to prove the following theorem.

**Theorem 8.1.** *Let  $\mathbf{k}$  be an algebraically closed field of characteristic 0, and let  $K^*/K$  be a finite extension of algebraic function fields of transcendence degree 2 over  $\mathbf{k}$ . Let  $\nu^*$  be a  $\mathbf{k}$ -valuation of  $K^*$ , with valuation ring  $V^*$  and value group  $\Gamma^*$ , and let  $\nu$  be the restriction of  $\nu^*$  to  $K$ , with valuation ring  $V$  and value group  $\Gamma$ . Suppose that  $R \rightarrow S$  is an extension of algebraic regular local rings with quotient fields  $K$  and  $K^*$  respectively, such that  $V^*$  dominates  $S$  and  $S$  dominates  $R$ . Then there exist sequences of quadratic transforms  $R \rightarrow \bar{R}$  and  $S \rightarrow \bar{S}$  along  $\nu^*$  such that  $\bar{S}$  dominates  $\bar{R}$  and the map between generating sequences of  $\nu$  and  $\nu^*$  in  $\bar{R}$  and  $\bar{S}$  respectively, has a toroidal structure.*

The lemma below is crucial in the proof of the theorem.

**Lemma 8.2.** *In the set up of Theorem 8.1, assume that  $\Gamma^*$  is a subgroup of  $\mathbb{Q}$  and  $V^*/m_{V^*} = \mathbf{k}$ . Suppose that  $R$  has regular parameters  $(u, v)$  and  $S$  has regular parameters  $(x, y)$  such that*

$$\begin{aligned} u &= x^t \delta \\ v &= y, \end{aligned} \tag{8.1}$$

where  $t$  is a positive integer and  $\delta$  is a unit in  $S$ . Let  $p$  and  $q$  be positive coprime integers such that  $\frac{\nu^*(y)}{\nu^*(x)} = \frac{p}{q}$  and let  $k = \epsilon(p, q)$ . Let  $\bar{p}$  and  $\bar{q}$  be positive coprime integers such that  $\frac{\nu(v)}{\nu(u)} = \frac{\bar{p}}{\bar{q}}$  and let  $\bar{k} = \epsilon(\bar{p}, \bar{q})$ . Let  $g$  be the greatest common divisor of  $t$  and  $p$ .

Then the sequences of quadratic transforms  $R = R_0 \rightarrow R_1 \rightarrow \dots \rightarrow R_{\bar{k}}$  and  $S = S_0 \rightarrow S_1 \rightarrow \dots \rightarrow S_k$  along  $\nu^*$  satisfy the following property:  $R_{\bar{k}}$  and  $S_k$  are free rings and there exist permissible systems of regular parameters  $(U, V)$  in  $R_{\bar{k}}$  and  $(X, Y)$  in  $S_k$  such that

$$\begin{aligned} U &= X^g \Delta \\ V &= Y \end{aligned} \tag{8.2}$$

for some unit  $\Delta \in S_k$ .

*Proof.* We notice first that  $\frac{\nu(v)}{\nu(u)} = \frac{\nu^*(y)}{t\nu^*(x)} = \frac{p}{tq}$ . Writing  $t = gt'$  and  $p = gp'$ , where  $(t', p') = 1$ , gives  $\bar{p} = p'$  and  $\bar{q} = qt'$ .

Let  $a$  and  $b$  be nonnegative integers such that  $a \leq p$ ,  $b < q$  and  $aq - bp = 1$ . Let  $\bar{a}$  and  $\bar{b}$  be nonnegative integers such that  $\bar{a} \leq \bar{p}$ ,  $\bar{b} < \bar{q}$  and  $\bar{a}\bar{q} - \bar{b}\bar{p} = 1$ .

By Lemma 6.1 applied to  $S$  and  $R$  respectively, we get that  $S_k$  has a permissible system of parameters  $(X, Y') = \left( \frac{x^a}{y^b}, \frac{y^q}{x^p} - c \right)$ , where  $c \in \mathbf{k}$  is the residue of  $\frac{y^q}{x^p}$ , and  $R_{\bar{k}}$  has a permissible system of parameters  $(U, V) = \left( \frac{u^{\bar{a}}}{v^{\bar{b}}}, \frac{v^{\bar{q}}}{u^{\bar{p}}} - \bar{c} \right)$ , where  $\bar{c} \in \mathbf{k}$  is the residue of  $\frac{v^{\bar{q}}}{u^{\bar{p}}}$ . Moreover,  $x = X^q(Y' + c)^b$ ,  $y = X^p(Y' + c)^a$  and  $u = U^{\bar{q}}(V + \bar{c})^{\bar{b}}$ ,  $v = U^{\bar{p}}(V + \bar{c})^{\bar{a}}$ .

Now

$$U = \frac{u^{\bar{a}}}{v^{\bar{b}}} = \frac{x^{t\bar{a}}}{y^{\bar{b}}} = \frac{[X^q(Y' + c)^b]^{t\bar{a}}}{[X^p(Y' + c)^a]^{\bar{b}}} = \frac{X^{qt\bar{a}}}{X^{p\bar{b}}}(Y' + c)^{bt\bar{a} - a\bar{b}} = X^g \Delta,$$

where  $\Delta = (Y' + c)^{bt\bar{a} - a\bar{b}}$ . Notice that the last equality holds since  $qt\bar{a} - p\bar{b} = qgt'\bar{a} - gp'\bar{b} = g(qt'\bar{a} - p'\bar{b}) = g(\bar{q}\bar{a} - \bar{p}\bar{b}) = g$ .

Furthermore notice that  $\frac{v^{\bar{q}}}{u^{\bar{p}}} = \frac{y^{\bar{q}}}{x^{t\bar{p}}} = \frac{y^{qt'}}{x^{gt'p'}}$ . Therefore  $\bar{c} = c^{t'}$ , and

$$V = \frac{v^{\bar{q}}}{u^{\bar{p}}} - \bar{c} = \left( \frac{y^q}{x^p} \right)^{t'} - c^{t'} = \left( \frac{y^q}{x^p} - c \right) \delta_2 = Y' \delta_2,$$

where  $\delta_2 = \frac{\left( \frac{y^q}{x^p} \right)^{t'} - c^{t'}}{\left( \frac{y^q}{x^p} - c \right)}$  is a unit in  $S_k$ . We set  $Y = Y' \delta_2$  to complete the proof.  $\square$

**Lemma 8.3.** *In the set up of Theorem 8.1, assume that  $\Gamma^*$  is a subgroup of  $\mathbb{Q}$  and  $V^*/m_{V^*} = \mathbf{k}$ . Suppose that  $R$  has regular parameters  $(u, v)$  and  $S$  has regular parameters  $(x, y)$  such that*

$$\begin{aligned} u &= x^t \delta \\ v &= y, \end{aligned} \tag{8.3}$$

where  $t$  is a positive integer and  $\delta$  is a unit in  $S$ .

Let  $\tilde{S} = S[\delta^{1/t}]_{m_{V^*} \cap S[\delta^{1/t}]}$  and let  $\{T_i\}_{i \geq 0}$  be the sequence of jumping polynomials in  $\tilde{S}$  such that  $T_0 = \tilde{x} = x\delta^{1/t}$  and  $T_1 = y$ . Then either  $\{u, \{T_i\}_{i > 0}\}$  is a sequence of jumping polynomials in  $R$  or there exist sequences of quadratic transforms  $R \rightarrow R'$  and  $S \rightarrow S'$  such that  $R'$  has a system of regular parameters  $(U, V)$ ,  $S'$  has a system of regular parameters  $(X, Y)$  and

$$\begin{aligned} U &= X^g \Delta \\ V &= Y, \end{aligned}$$

where  $g < t$  is a positive integer and  $\Delta$  is a unit in  $S'$ .

*Proof.* Without loss of generality we may assume that  $\Gamma^*$  is normalized so that  $\nu^*(x) = 1$ . Then  $\nu^*(\tilde{x}) = 1$  and  $\nu(u) = t$ .

For all  $i > 0$  let  $p_i$  and  $q_i$  be coprime integers defined in the construction of jumping polynomials  $\{T_i\}_{i \geq 0}$  in  $\tilde{S}$ . Denote by  $M = \min\{i > 0 \mid t \nmid p_i\}$ . We assume first that  $M = \infty$ , that is  $p_i$  is multiple of  $t$  for every  $i$ . Then since  $u = \tilde{x}^t$ , by Theorem 7.1 we get that  $\{u, \{T_i\}_{i > 0}\}$  is a sequence of jumping polynomials in  $R$ .

Assume now that  $M < \infty$ . Since  $u = \tilde{x}^t$  and  $t \mid p_i$  for all  $i < M$ , by Theorem 7.1 we get that  $\{u, \{T_i\}_{i=1}^M\} = \{T'_i\}_{i=0}^M$  is the beginning of a sequence of jumping polynomials in  $R$  and for all  $i < M$  the pairs of coprime integers  $(p'_i, q'_i)$  defined in the construction of the sequence  $\{T'_i\}_{i \geq 0}$  are  $(p'_i, q'_i) = (\frac{p_i}{t}, q_i)$ .

Recall that the integers  $k_i$  are defined as  $k_0 = 0$  and  $k_i = k_{i-1} + \epsilon(p_i, q_i)$  if  $i > 0$ . Let  $k'_0 = 0$  and  $k'_i = k'_{i-1} + \epsilon(p'_i, q'_i)$  for all  $i > 0$ . We will show first that the sequences of quadratic transforms

$$S = S_0 \rightarrow \dots \rightarrow S_{k_1} \rightarrow \dots \rightarrow S_{k_{M-1}}$$

and

$$R = R_0 \rightarrow \dots \rightarrow R_{k'_1} \rightarrow \dots \rightarrow R_{k'_{M-1}}$$

have the following property: for all  $0 \leq i \leq M-1$  the rings  $R_{k'_i}$  and  $S_{k_i}$  are free, there exist permissible systems of parameters  $(u_i, v_i)$  in  $R_{k'_i}$  and  $(x_i, y_i)$  in  $S_{k_i}$  and a unit  $\delta_i \in S_{k_i}$  such that

$$u_i = x_i^t \delta_i, \quad v_i = y_i$$

and

$$\nu^*(x_i) = \frac{1}{Q_i}, \quad \nu^*(y_i) = \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

The statement is trivial for  $i = 0$ . Assume that  $i > 0$  and that the statement holds for  $i-1$ . Then Lemma 8.2 applies to  $R_{k'_{i-1}} \subset S_{k_{i-1}}$ . We notice that  $\frac{\nu^*(y_{i-1})}{\nu^*(x_{i-1})} = \frac{p_i}{q_i}$  and  $\frac{\nu(v_{i-1})}{\nu(u_{i-1})} = \frac{p_i}{tq_i} = \frac{p'_i}{q'_i}$ , and therefore  $k = \epsilon(p_i, q_i)$  and  $\bar{k} = \epsilon(p'_i, q'_i)$ . Thus  $R_{k'_i}$  and  $S_{k_i}$  are free rings and there exist permissible systems of regular parameters  $(u_i, w_i)$  in  $R_{k'_i}$  and  $(x_i, z_i)$  in  $S_{k_i}$  such that  $u_i = x_i^t \delta_i$  and  $w_i = z_i$  for some unit  $\delta_i \in S_{k_i}$ .

Now by Theorem 7.5 applied to  $R$  with  $\nu$  replaced by  $\tilde{\nu} = \frac{1}{t}\nu$  we get that  $R_{k'_i}$  has a system of regular parameters  $(h_i, v_i)$  such that  $h_i$  is an exceptional parameter,

$$\nu(h_i) = t\tilde{\nu}(h_i) = t \frac{1}{q'_1 \cdots q'_i} = \frac{t}{q_1 \cdots q_i} = \frac{t}{Q_i}$$

and

$$\nu(v_i) = t\tilde{\nu}(v_i) = t \frac{1}{q'_1 \cdots q'_i} \cdot \frac{p'_{i+1}}{q'_{i+1}} = \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}.$$

Since  $u_i$  is also an exceptional parameter in  $R_{k'_i}$  we have  $u_i = h_i \gamma$  for some unit  $\gamma \in R_{k'_i}$ . Therefore  $(u_i, v_i)$  form a permissible system of parameters in  $R_{k'_i}$  and  $\nu(u_i) = \frac{t}{Q_i}$ . Notice also that  $v_i = \alpha u_i + \beta w_i$ , where  $\alpha, \beta \in S_{k_i}$ . Moreover,  $\beta$  is a unit in  $R_{k'_i}$ , since the image of  $v_i$  is a regular parameter in  $R_{k'_i}/(u_i)$ . This implies

that  $v_i = \alpha x^t \delta_i + \beta z_i$  is also a regular parameter in  $S_{k_i}$  and  $(x_i, v_i)$  form a permissible system of parameters in  $S_{k_i}$ . We set  $y_i = v_i$  and observe that  $\nu^*(x_i) = \frac{1}{t} \nu(u_i) = \frac{1}{Q_i}$  and  $\nu^*(y_i) = \nu(v_i) = \frac{1}{Q_i} \cdot \frac{p_{i+1}}{q_{i+1}}$ .

To finish the proof of the lemma we apply Lemma 8.2 to  $R_{k'_{M-1}} \subset S_{k_{M-1}}$ . We have  $p = p_M$ ,  $q = q_M$  and  $\bar{p} = p'_M$ ,  $\bar{q} = q'_M$ . Thus  $R' = R_{k'_M}$  has regular parameters  $(U, V)$  and  $S' = S_{k_M}$  has regular parameters  $(X, Y)$  such that

$$\begin{aligned} U &= X^g \Delta \\ V &= Y, \end{aligned}$$

where  $\Delta$  is a unit in  $S_{k_M}$  and  $g = (p_M, t) < t$ . □

We are now ready to prove Theorem 8.1.

*Proof.* By the discussion of Section 3 we only need to consider the case when  $\Gamma^*$  is a subgroup of  $\mathbb{Q}$  and  $\text{trdeg}_{\mathbb{k}}(V^*/m_{V^*}) = 0$ . Then by the Strong Monomialization theorem we may assume that there exist regular parameters  $(u, v)$  in  $R$  and  $(x, y)$  in  $S$  such that  $u = x^t \delta$  and  $v = y$  for some unit  $\delta \in S$ . If  $t = 1$  then  $R = S$  and the conclusion of the theorem is trivial, so assume that  $t > 1$ .

We set  $\tilde{S} = S[\delta^{1/t}]_{m_{V^*} \cap S[\delta^{1/t}]}$  and  $\tilde{x} = x\delta^{1/t}$ . Let  $\{T_i\}_{i \geq 0}$  be a sequence of jumping polynomials in  $\tilde{S}$  such that  $T_0 = \tilde{x}$  and  $T_1 = y$ . For all  $i > 0$  let the coprime integers  $p_i$  and  $q_i$  be defined as in the construction of jumping polynomials  $\{T_i\}_{i \geq 0}$  in  $\tilde{S}$ .

First, let us assume that  $\{u, \{T_i\}_{i > 0}\}$  is a sequence of jumping polynomials in  $R$ . By the proof of Lemma 8.3 we have that  $t | p_i$  for all  $i > 0$ .

If  $\Gamma^*$  is a discrete subgroup of  $\mathbb{Q}$ , then  $\Gamma$  is also a discrete subgroup of  $\mathbb{Q}$ . We may further assume that  $\nu^*(x)$  generates  $\Gamma^*$  and  $\nu(u)$  generates  $\Gamma$ . Then by Theorem 4.5 we get that  $\{u, \{T_i\}_{i > 0}\}$  is a generating sequence in  $R$ . Also since for all  $i > 0$  the inclusion  $T_i \in \tilde{S}$  holds and  $\beta_i = \nu^*(T_i)$ , by Corollary 5.4 we see that  $q_i = 1$  for all  $i > 0$ . Now repeating the proof of Theorem 4.5 we get that  $\{x, \{T_i\}_{i > 0}\}$  is a generating sequence of  $S$ .

If  $\Gamma^*$  is a non-discrete subgroup of  $\mathbb{Q}$ , then  $\Gamma$  and the value group of  $QF(\tilde{S})$  are also non-discrete subgroups of  $\mathbb{Q}$ . If  $\{H_i\}_{i \geq 0}$  is the sequence of independent jumping polynomials in  $\tilde{S}$ , then  $\{u, \{H_i\}_{i > 0}\}$  is a sequence of independent jumping polynomials in  $R$ . By Theorem 4.6 we have that  $\{H_i\}_{i \geq 0}$  is a generating sequence in  $\tilde{S}$ . Moreover, it is minimal since  $p_1 > 1$  as a multiple of  $t > 1$ . This implies that  $\{x, \{H_i\}_{i > 0}\}$  is a minimal generating sequence in  $S$ . Also by Theorem 4.6 we have that  $\{u, \{H_i\}_{i > 0}\}$  is a generating sequence in  $R$ .

If  $\{u, \{T_i\}_{i > 0}\}$  is not a sequence of jumping polynomials in  $R$ , let  $R'$ ,  $S'$  and  $g$  be as in the proof of Lemma 8.3. Notice that  $g < t$ , i.e., the exponent of Strong Monomialization has dropped. We now repeat the above argument starting with the

rings  $R'$  and  $S'$  (instead of  $R$  and  $S$ ). After a finite number of iterations we obtain the desired conclusion.  $\square$

**Remark 8.4.** In the proof of Theorem 8.1 we have  $\{u, \{H_i\}_{i>0}\}$  is a minimal generating sequence of  $\nu$  in  $R$  if  $p_1 \neq t$ , otherwise  $\{H_i\}_{i>0}$  is a minimal generating sequence of  $\nu$  in  $R$ .

**Remark 8.5.** Assumptions and notations are as in the statement of Theorem 8.1. By [5, Theorem 6.1] there exist sequences of quadratic transforms  $R \rightarrow \bar{R}$ ,  $S \rightarrow \bar{S}$  along  $\nu^*$  such that  $\bar{R}$  has regular parameters  $(u, v)$ ,  $\bar{S}$  has regular parameters  $(x, y)$  such that

$$\begin{aligned} u &= x^e \delta \\ v &= y \end{aligned} \tag{8.4}$$

where  $\delta$  is a unit in  $\bar{S}$ , and  $e = [\Gamma^* : \Gamma]$  is the ramification index of  $\nu^*$  relative to  $\nu$ . The monomial form of (8.4) is preserved by the sequences of quadratic transforms of the proof of Theorem 8.1. Furthermore, the exponent  $e$  does not drop under such sequences of quadratic transforms (see the proof of [5, Theorem 6.1]). It follows from the proof of Theorem 8.1 that the map between generating sequences of  $\nu$  and  $\nu^*$  in  $\bar{R}$  and  $\bar{S}$  respectively, has the desired toroidal structure.

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