EULER AND BERNOULLI NUMBERS AND $\zeta(s)$

Tewodros Amdeberhan

Department of Mathematics, Temple University, Philadelphia PA 19122, USA
tewodros@euclid.math.temple.edu

We recall that the **'even-index' Bernoulli numbers**, B_{2n} are defined by the generating function relation:

$$\frac{x}{e^x-1}+\frac{1}{2}x=\sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}x^{2n}, \tag{1-e}$$

and the **'odd-index' Euler numbers**, E_{2n+1} by the relation:

$$x sec(x)=\sum_{n=0}^{\infty}\frac{E_{2n+1}}{(2n+1)!}x^{2n+1}. \tag{1-o}$$

Then, the following identities link the Bernoulli numbers with the **Reimann-Zeta function**, $\zeta(s)$. For the even-index Bernoulli numbers, we have

$$\frac{4^m-1}{4^m}\zeta(2m)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^{2m}}=(-1)^{m-1}\frac{4^m(4^m-1)B_{2m}}{2(2m)!}\left(\frac{\pi}{2}\right)^{2m}, \tag{2-e}$$

for integers, $m\geq 1$. Note also the connection that (2-e) has with the tangent function:

$$\frac{x}{2}tan(x)=\sum_{m=1}^{\infty}(-1)^{m-1}\frac{4^m(4^m-1)B_{2m}}{2(2m)!}x^{2m}.$$

Similarly, for the odd-index Euler numbers we have

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{2m+1}}=\frac{E_{2m+1}}{2(2m+1)!}\left(\frac{\pi}{2}\right)^{2m+1}, \tag{2-o}$$

for integers, $m\geq 0$. We also get,

$$\sum_{n=0}^{\infty}\frac{1}{(4n+1)^{2m+1}}=\frac{E_{2m+1}}{4(2m+1)!}\left(\frac{\pi}{2}\right)^{2m+1}+\frac{2^{2m+1}-1}{2^{2m+2}}\zeta(2m+1). \tag{3}$$

Typeset by $\mathcal{AMS}$-TEX

$$\binom{k+1}{1}B_k + \binom{k+1}{2}B_{k-1} + \cdots + \binom{k+1}{k}B_1 + B_0 = 0; \tag{4-e}$$

where $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_{2n+1} = 0$ for $n > 0$.

$$(-1)^m\binom{2m+1}{1}E_1 + (-1)^{m-1}\binom{2m+1}{3}E_3 + \cdots - \binom{2m+1}{2m-1}E_{2m-1} + E_{2m+1} = 0; \tag{4-o}$$

where $E_1 = 1$, and $E_{2n} = 0$ for $n \geq 0$.

NOTE: From (4-e) and (4-o), we can see that B_{2n} are *rational numbers,* while $\frac{E_{2n+1}}{2n+1}$ are *positive integers.* We also have the following inter-relation:

$$\sum_{j=1}^{n}(-1)^{j-1}4^j(4^j-1)\binom{2n+1}{2j}B_{2j}E_{2n-2j+1} = 2nE_{2n+1}. \tag{5}$$

H. Chen [1] rediscovers Euler's formula $\zeta(2) = \frac{\pi^2}{6}$, using a simple transformation. Here, we give such transformations in more dimensions. Observe the appearance of Euler and Bernoulli numbers measuring deficiencies of these transformations in "fitting" a cube into a tetrahedron.

Denote $\square^m$:= m-dimensional unit cube, and the m-dimensional tetrahedron given by Δ^m:=$\{u | u_1 + u_2 + \cdots + u_m \leq \frac{\pi}{2}, u_i \geq 0\}$ and apply the transformation:

$$x_1 = \frac{sin(u_1)}{cos(u_2)}, x_2 = \frac{sin(u_2)}{cos(u_3)}, \ldots, x_m = \frac{sin(u_m)}{cos(u_1)}$$

on Δ^m, then denoting the resulting domain by R_m we find that

$$\int_{\Delta^{2m}} du = \int_{R_{2m}} \frac{dx}{1 - x_1^2 \ldots x_{2m}^2} = \frac{2(-1)^{m-1}}{4^m(4^m-1)B_{2m}} \int_{\square^{2m}} \frac{dx}{1 - x_1^2 \ldots x_{2m}^2}, \tag{6-e}$$

for $m \geq 1$. And also that,

$$\int_{\Delta^{2m+1}} du = \int_{R_{2m+1}} \frac{dx}{1 + x_1^2 \ldots x_{2m+1}^2} = \frac{2}{E_{2m+1}} \int_{\square^{2m+1}} \frac{dx}{1 + x_1^2 \ldots x_{2m+1}^2}, \tag{6-o}$$

for $m \geq 0$.

References

[1] H. Chen, *Euler's formula* $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ *by double integration*, Math. and Computer Educ. **30 #3** (1996), 295-297.