A one-parameter family of cylindrically symmetric solution approaching Einstein's universe

T. Amdeberhan[*]

DIMACS, Rutgers Univ.

96 Frelinghuysen Rd.

Piscataway, NJ 08854-8010

and

DeVry College of Technology, NJ, USA

amdberha@nj.devry.edu,

tewodros@math.temple.edu

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Abstract

In this note we describe a stationary cylindrically symmetric solution of Einstein's equation with matter consisting of a positive cosmological constant and an infinite cylinder of rotating dust. The solution approaches Einstein static universe solution. This result is complementary to Iftime [Class Quantum Grav 19 (2002)].

1 Introduction

The basic partial differential equations of general relativity are Einstein's field equations. In general these equations are essentially hyperbolic and are coupled to other partial differential equations describing the matter content of spacetime. A special feature of the Einstein equations is that initial data cannot be given freely. They must satisfy constraint equations. To prove existence theorems, it is necessary to show the existence of a solution of the constraints, and the usual method here relies on the theory of elliptic equations.

The local existence theory of solutions of the Einstein equations is rather well understood. On the other hand, the problem of proving *global* existence theorems is totally another matter (an important aspect and the depth of existence theorems in general relativity which one should be aware of is their relation to the cosmic censorship hypothesis, see cf.[6].) Progress is made possible by focusing on simplified models. The most common simplifications are to look at solutions with various types of *axial-symmetry* and solutions for *small data*. The most extensive results on global inhomogeneous solutions of the Einstein equations up to now concern spherically symmetric solutions.

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[*]this work was done during the author's stay at DIMACS

At present we concentrate on yet another breed of symmetry of solutions: cylindrical. More precisely, we study a spacetime satisfying Einstein's field equations with positive cosmological constant, describing a dust cylinder in non-rigid rotation, which approaches Einstein's cosmological static universe on the axis of rotation. This stationary cylindrically symmetric solutions result parallels and is complementary to that of Iftime [3].

The problem of Einstein's equation without cosmological constant and with negative cosmological constant have already appeared in the literature. In [1], a vacuum stationary cylindrically symmetric solution with negative cosmological constant is matched to an interior rotating dust cylinder cut out of a Gödel universe, whose metric might read as

$$\mathrm{d}s^2 = \mathrm{d}R^2 + \mathrm{d}Z^2 + 4h^2(\sinh^2\rho - \sinh^4\rho)\mathrm{d}\psi^2 - 4(2)^{\frac{1}{2}}h\sinh^2\rho\,\mathrm{d}\psi\,\mathrm{d}T - \mathrm{d}T^2.$$

Van Stockum [7] found a rigidly rotating infinitely long dust cylinder without cosmological constant which has various exterior metrics.

The spatially closed, static Einstein universe in usual form,

$$
(1) \qquad \mathrm{d}s_E^2 = \mathrm{d}\eta^2 + \sin^2\eta(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) - c^2\mathrm{d}\psi^2
$$
$$
\varphi \in [0, 2\pi], \quad \eta \in [0, \pi], \quad \theta \in [0, \pi], \quad \psi \in \mathrm{R}
$$

is the simplest cosmological dust model with constant curvature $K = \mathrm{const}$ and positive cosmological constant $\Lambda = \mathrm{const}$, $\Lambda > 0$. The field is produced by a energy-momentum tensor T_{ab} of perfect-fluid:

$$
(2) \qquad \kappa T_{ab} = -\Lambda g_{ab} + \mu u_a u_b, \quad \mu > 0, \quad \Lambda = \mathrm{const.} > 0.
$$

where $\Lambda = \frac{1}{K^2}$ and $\mu = \frac{2}{K^2} = 2\Lambda = const.$. The Einstein metric in cylindrical coordinates [4] will be used vigorously:

$$
(3) \qquad \mathrm{d}s^2 = e^{2V_0(r)}(\mathrm{d}r^2 + \mathrm{d}z^2) + W_0^2(r)\mathrm{d}\varphi^2 - \mathrm{d}t^2
$$

where $W_0(r)$ and $V_0(r)$ have the form :

$$
(4) \qquad V_0(r) = \frac{1}{2}\ln\left(\gamma - \lambda^2\left(\frac{1 - e^{2\lambda(r-\nu)}}{1 + e^{2\lambda(r-\nu)}}\right)^2\right) - \ln\sqrt{\Lambda}
$$

$$
(5) \qquad W_0(r) = \frac{1 - e^{2\lambda(r-\nu)}}{\left(1 - e^{2\lambda(r-\nu)}\right)^{\frac{\gamma}{\lambda^3}}}\, e^{2\lambda(r-\nu)(\frac{1}{2} - \frac{\gamma}{2\lambda^3})}
$$

and γ, α, $\lambda \neq 0$ and ν are constants of integration and $\mu = 2\Lambda = \mathrm{const}$, the dust density, respectively.

The space-time of special relativity is described mathematically by the Minkowski space (M, η). It has been shown that Minkowski spacetime is conformal to a finite region of the Einstein static universe. The de Sitter space-times are also conformal to a finite part of the Einstein universe $\mathrm{d}s_E^2$ and generally, all the closed Robertson-Walker metrics, including Minkowski spacetime and the Sitter spacetimes as special cases, are conformally equivalent to the Einstein static universe [2].

2 The metric

Stationary gravitational fields are characterized by the existence of a timelike Killing vector field ξ. Therefore in a stationary space-time (M, g) we can construct a global causal structure. In other words we can introduce a coordinate system $(x^a) = (x^\alpha, t)$ with $\xi = \frac{\partial}{\partial t}$. The metric g_{ab} in these coordinates is independent of t and has, in general, the following form:

$$(6) \qquad \mathrm{d}s^2 = h_{\alpha\beta}\mathrm{d}x^\alpha\mathrm{d}x^\beta + F(\mathrm{d}t + A_\alpha\mathrm{d}x^\alpha)^2, \qquad F \equiv \xi_a\xi^a < 0$$

The unitary timelike vector field $h^0 \equiv (-F)^{-\frac{1}{2}}\xi$ is globally defined on M; it indicates the time-orientation in every point $p \in M$ and gives a global time coordinate t on M [5]. Stationarity (i.e. time translation symmetry) implies that there exists a 1-dimensional group G_1 of isometries ϕ_t whose orbits are timelike curves parameterized by t. Using the 3-projection formalism (developed by Geroch 1971) of a four-dimensional spacetime manifold (M, g) onto the three-dimensional differentiable factor manifold $\mathcal{S}_3 = M/G_1$, the Einstein's field equations

$$(7) \qquad R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab},$$

for stationary fields take the following simplified form:

$$(8)\begin{cases} R_{ab}^{(3)} = \dfrac{1}{2}F^{-2}(\dfrac{\partial F}{\partial x^a}\dfrac{\partial F}{\partial x^b} + \omega_a\omega_b) + \kappa(h_a^c h_b^d - F^{-2}\tilde{h}_{ab}\xi^a\xi^b)(T_{cd} - \dfrac{1}{2}Tg_{cd}); \\[2mm] F_{,a}^{\|a} = F^{-1}\tilde{h}_{ab}(\dfrac{\partial F}{\partial x^a}\dfrac{\partial F}{\partial x^b} - \omega_a\omega_b) - 2\kappa F^{-1}\xi^a\xi^b(T_{ab} - \dfrac{1}{2}Tg_{ab}); \\[2mm] \omega_a^{\|a} = 2F^{-1}\tilde{h}_{ab}\dfrac{\partial F}{\partial x^a}\omega_b \\[2mm] F\epsilon^{abc}\omega_{c,b} = 2\kappa h_b^a T_c^b \xi^c \end{cases}$$

Here, "$\|$" denotes the covariant derivative associated with the conformal metric tensor $\tilde{h}_{ab} = -Fh_{ab}$ on $\mathcal{S}_3$ ($h_{ab} = g_{ab} + h_a^0 h_b^0$ is the projection tensor) and $\omega^a = \frac{1}{2}\epsilon^{abcd}\xi_b\xi_{c;d} \neq 0$ [1] is the rotational vector ($\omega^a\xi_a = 0, \quad \mathcal{L}_\xi\omega = 0$). We shall consider that the metric g_{ab} has a cylindrical symmetry, i.e. it admits as well an Abelian group of isometries G_2 generated by two spacelike Killing vector fields η and ζ, $\mathcal{L}_\eta g_{ab} = \mathcal{L}_\zeta g_{ab} = 0, \quad \eta_a\eta^a > 0, \quad \zeta_a\zeta^a > 0$ and the integral curves of η are closed (spatial) curves. We are using Kundt's theorem which states that an axisymmetric metric can be written in a (2+2)-split if and only if the conditions

$$(9) \qquad (\eta^{[a}\xi^b\xi^{c;d]})_{;e} = 0 = (\xi^{[a}\eta^b\eta^{c;d]})_{;e}$$

are satisfied. The existence of the orthogonal 2-surfaces is assured for the dust solutions, provided that the 4-velocity of dust satisfies the condition

$$(10) \quad u_{[a}\xi_b\eta_{c]} = 0, \quad u^a = (-H)^{-\frac{1}{2}}(\xi^a + \Omega\eta^a) = (-H)^{-\frac{1}{2}}l^i\xi_i^a, \quad \text{where}$$

$$l^i \equiv (1, \Omega), \quad H = \gamma_{ij}l^il^j, \quad \gamma_{ij} \equiv \xi_i^a\xi_{aj}, \quad i, j = 1, 2, \quad \xi_1 = \xi; \; \xi_2 = \eta$$

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[1]Here we use the convention: round brackets denote symmetrization and square brackets antisymmetrization and Ω is the angular velocity.

in other words, if the trajectories of the dust lie on the transitivity surfaces of the group generated by the Killing vectors ξ, η. In what follows, we will assume that this is true. Using an adapted coordinate system, the metric (6) can be written in standard form

$$(11) \qquad \mathrm{d}s^2 = e^{-2U}[e^{2V}(\mathrm{d}r^2 + \mathrm{d}z^2) + W^2\mathrm{d}\varphi^2] - e^{2U}(\mathrm{d}t + A\mathrm{d}\varphi)^2$$

where the functions [2] U, V, W and A depend only on the coordinates (r, z); these coordinates are also conformal flat coordinates on the 2-surface S_2 orthogonal to 2-surface T_2 of the commuting Killing vectors $\xi = \partial_t$ and $\eta = \partial_\varphi$. If we identify the 4-velocity of the dust u^a with timelike Killing vector $\xi^a = \partial_t = (0, 0, 0, 1)$ then (11) represents a co-moving system ($x^1 = r$, $x^2 = z$, $x^3 = \varphi$, $x^0 = t$) with dust, $u_a = \xi_a = (0,\, 0,\, -e^{2U}A,\, -e^{2U})$ and

$$(12) \qquad \begin{cases} g_{11} = g_{22} = e^{-2U+2V} = h_{11} = h_{22}, \\ g_{33} = e^{-2U}W^2 - e^{2V}A^2 = h_{33}, \quad g_{00} = \xi_0 = -e^{2U} = F, \\ g_{03} = \xi_3 = -e^{2U}A, \quad g_{13} = g_{23} = g_{10} = g_{20} = 0. \end{cases}$$

We can use the complex coordinates $(q, \bar{q})$ on the 2-surface S_2:

$$(13) \qquad q = \frac{1}{\sqrt{2}}(r + iz)$$

and the stationary axisymmetric metric (11) takes the Lewis- Papapetrou form

$$(14) \qquad \mathrm{d}s^2 = e^{-2U}(e^{2V}\mathrm{d}q\mathrm{d}\bar{q} + W^2\mathrm{d}\varphi^2) - e^{2U}(\mathrm{d}t + A\mathrm{d}\varphi)^2$$

The surface element on T_2 is $f_{ab} = 2\xi_{[a}\eta_{b]}$, $f_{ab}f^{ab} < 0$ and the surface element on S_2 is $\tilde{f}_{ab}$, the dual tensor of f_{ab}, $\tilde{f}_{ab} = \frac{1}{2}\epsilon_{abcd}f^{cd}$. Thus the Einstein's dust equations with cosmological constant $\Lambda > 0$ (8) for the metric (12) will take the following form:

$$(15) \quad \begin{cases} \dfrac{\partial^2 W}{\partial q \partial \bar{q}} = -\Lambda W e^{2V-2U} \\[2mm] \dfrac{\partial^2 U}{\partial q \partial \bar{q}} + \dfrac{1}{2W}\left(\dfrac{\partial U}{\partial q}\dfrac{\partial W}{\partial \bar{q}} + \dfrac{\partial U}{\partial \bar{q}}\dfrac{\partial W}{\partial q}\right) + \dfrac{1}{2W^2}e^{4U}\dfrac{\partial A}{\partial q}\dfrac{\partial A}{\partial \bar{q}} = (\mu - 2\Lambda)\dfrac{e^{2V-2U}}{4} \\[2mm] \dfrac{\partial^2 A}{\partial q \partial \bar{q}} - \dfrac{1}{2W}\left(\dfrac{\partial A}{\partial q}\dfrac{\partial W}{\partial \bar{q}} + \dfrac{\partial A}{\partial \bar{q}}\dfrac{\partial W}{\partial q}\right) + 2\left(\dfrac{\partial A}{\partial q}\dfrac{\partial U}{\partial \bar{q}} + \dfrac{\partial A}{\partial \bar{q}}\dfrac{\partial U}{\partial q}\right) = 0 \\[2mm] \dfrac{\partial^2 W}{\partial q \partial \bar{q}} - 2\dfrac{\partial W}{\partial q}\dfrac{\partial V}{\partial q} + 2W\left(\dfrac{\partial U}{\partial q}\right)^2 - \dfrac{1}{2W}e^{4U}\left(\dfrac{\partial A}{\partial q}\right)^2 = 0 \\[2mm] \dfrac{\partial^2 V}{\partial q \partial \bar{q}} + \dfrac{\partial U}{\partial q}\dfrac{\partial U}{\partial \bar{q}} + \dfrac{1}{(2W)^2}e^{4U}\dfrac{\partial A}{\partial q}\dfrac{\partial A}{\partial \bar{q}} = -\Lambda\dfrac{e^{2V-2U}}{2} \end{cases}$$

Here $\Delta = \dfrac{\partial^2}{\partial r^2} + \dfrac{\partial^2}{\partial z^2} = 2\dfrac{\partial^2}{\partial q \partial \bar{q}}$ is the Laplace operator and the energy-momentum tensor T_{ab} has the form (2) with constant $\Lambda > 0$ and $\mu(r) > 0$.

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[2]The function W is defined invariantly as $W^2 \equiv -2\xi_{[a}\eta_{b]}\xi^a\eta^b$.

The conservation law $T^{ab}_{;b} = 0$ implies $U_{,a} = 0$. We obtain then $U = constant$ as a consequence of the field equations and it will be used in the place of one of the Einstein's equations. Assuming that $U = 0$ in the expressions of the metric functions (12) we obtain that the matter current paths are geodesics ($\dot{u}_a = u_{a;b}u^b = 0$), without expansion ($\theta = u^a_{;a} = 0$), in a non-rigidly rotation ($\omega = \sqrt{\frac{1}{2}\omega_{ab}\omega^{ab}} \neq 0$) and with $\sigma \neq 0$. Also, by taking into account the third symmetry (the presence of the spacelike Killing vector field $\zeta = \partial_z$) and for the metric

$$(16) \qquad \mathrm{d}s^2 = e^{2V(r)}(\mathrm{d}r^2 + \mathrm{d}z^2) + W^2(r)\mathrm{d}\varphi - (\mathrm{d}t + A(r)\mathrm{d}\varphi)^2$$

we can reduce the field equations (15) to the ordinary differential equations

$$(17) \qquad \begin{cases} W'' = -2\Lambda W e^{2V} \\ 2A'^2 = (\mu - 2\Lambda)W^2 e^{2V} \\ \dfrac{A''}{A'} = 2\dfrac{W'}{W} \\ W'' - 4W'V' - \dfrac{1}{W}A'^2 = 0 \\ V'' + \dfrac{1}{2W^2}A'^2 = -\Lambda e^{2V} \end{cases}$$

in the unknown metric functions $V(r)$, $W(r)$, $A(r)$ and $\mu(r)$, where we denoted $\frac{\partial}{\partial r} = '$.

The system of equations (17) can be further compactified to the following form:

$$(18) \qquad \begin{cases} \dfrac{W''}{W} = -2\Lambda e^{2V} \\ A' = aW^2 \\ V' = bW^2 \\ (\mu - 2\Lambda)e^{2V} = 2a^2 W^2 \\ W'' - 4bW'W^2 - a^2 W^3 = 0 \end{cases}$$

where $a \neq 0$, $b \neq 0$ are positive constants and Λ is the positive cosmological constant. The system (18) does not have an explicit analytical solution for $W(r)$, $V(r)$, $A(r)$ and $\mu(r)$ as functions of radius r. Therefore we shall look forward to derive a good approximation of the solution and this we accomplish by "fattening" Einstein's solutions along a narrow tube $(r, b; a)$. We remark from the form of the system, that we are looking for a one-parameter[3] family $g_{ij}(b)$ of solutions, where b measures the size of perturbation, in the sense that $g_{ij}(b)$ are differentiable in b, and for $b = 0$ we obtain Einstein universe.

In what follows we shall show that the solution of (18) is approaching Einstein universe solution $g_{ij}(r, 0)$, as radius r goes to zero:

$$(19) \qquad g_{ij}(r, b) = g_{ij}(r, 0) + bg_{ij,b}(r, 0) + \frac{b^2}{2}g_{ij,bb}(r, 0) + ...$$

We shall perturb the solution as power series in b about Einstein universe solution and give a good approximation to $g_{ij}(b)$ for sufficiently small b (note: a is free.) To do so we differentiate the system (18) with respect to b, then take b to be zero and obtain the following equations:

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[3]It is actually a two-parameter family of solutions $g_{ij}(a, b)$

$$(20) \quad \begin{cases} \dot{W}''(r,0) - 4W_0^2 W_0' - 3a^2 W_0^2 \dot{W}(r,0) = 0 \\ \dot{V}'(r,0) = W_0^2 \\ \dot{\mu}(r,0)e^{2V_0} = 0 \\ \dot{A}'(r,0) = 2aW_0\dot{W}(r,0) \end{cases}$$

for the functions $\dot{W}(r,0)$, $\dot{V}(r,0)$, $\dot{\mu}(r,0)$ $\dot{A}(r,0)$. We designated by $\frac{\partial}{\partial b} = \dot{}$, and $W_0(r)$, $V_0(r)$ are the metric functions of the Einstein universe.

By choosing appropriate constants of integration $\nu = 0$, $\lambda = 1$, $\gamma = 1$ in (5) we get $W_0(r) = 1$, [4]. Then the system (20) can be completely integrated and take the following form:

$$(21) \quad \begin{cases} \dot{W}(r,0) = [c_1\sinh(\sqrt{3}ar) + c_2\cosh(\sqrt{3}ar)]. \\ \dot{A}(r,0) = \dfrac{2}{\sqrt{3}}[c_1\cosh(\sqrt{3}ar) + c_2\sinh(\sqrt{3}ar)] \\ \dot{\mu}(r,a) = 0 \\ \dot{V}(r,0) = \dfrac{2b}{\sqrt{3}a}[c_1\cosh(\sqrt{3}ar) + c_2\sinh(\sqrt{3}ar)]. \end{cases}$$

Then $g_{ij}(r,0) + bg_{ij,b}(r,0)$ will give a good approximation to the solution of the Einstein field equations (18), $g_{ij}(r,b)$, for small b when r approaches the axis of rotation $\eta = 0$:

$$(22) \quad \begin{cases} W(r,b) = 1 + b[c_1\sinh(\sqrt{3}ar) + c_2\cosh(\sqrt{3}ar)] \\ A(r,b) = \dfrac{2b}{\sqrt{3}}[c_1\cosh(\sqrt{3}ar) + c_2\sinh(\sqrt{3}ar)] \\ \mu(r,b) = 2\Lambda \\ V(r,b) = V_0(r) + br + \dfrac{b^2}{\sqrt{3}a}[c_1\cosh(\sqrt{3}ar) + c_2\sinh(\sqrt{3}ar)]. \end{cases}$$

The resulting metric thus becomes an approximant around the axis of rotation and it depends on three parameters a, b and Λ.

3 Acknowledgments

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