THE MACMAHON q-CATALAN IS CONVEX

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Abstract. Let $n \geq 2$ be an integer. We prove the convexity of the so-called MacMahon q-Catalan polynomials $C_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$ viewed as functions of q over the entire set of reals. Along the way, several auxiliary properties of the q-Catalan polynomials and intermediate results in the form of inequalities are presented, with the aim to make the paper self-contained. We also include a commentary on the convexity of the generating function for the integer partitions.

1. Introduction

For $n \in \mathbb{N}$, let $[n]_q = \frac{1-q^n}{1-q} = 1+q+\cdots+q^{n-1}$ and $[n]!_q = [1]_q[2]_q \cdots [n]_q$. We adopt $[0]_q = 0$ and $[0]!_q = 1$. The current literature embraces different versions of the q-Catalan polynomials, among them is MacMahon's q-Catalan polynomial defined by

$$C_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = \frac{[2n]!_q}{[n+1]!_q\,[n]!_q},$$

with $C_0(q) = C_1(q) = 1$. For contrast, several authors investigated the *Gaussian polynomials* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ for symmetry, unimodality [7] and log-concavity [3]. However, the above q-Catalan polynomials are symmetric (palindromic) but they do not enjoy the other properties. There are some combinatorial interpretations of MacMahon's q-Catalan in the context of the *maj* statistic [6], simultaneous core-partitions [2], [5] and a slightly altered concept of *parity unimodality* [8]. Our main goal, in this paper, is to explore another interesting question which incidentally seems to have been overlooked so far: the q-Catalan polynomials are *strictly convex functions* of q; that means, $C_n''(q) > 0$ for $n \geq 2$. This conjecture is due to William Y. C. Chen, from August 2015, who also posited that all even-order derivatives of $C_n(q)$ are positive [4]. We thank him for bringing his question to our attention. In the present work, we are able to produce the first complete proof for the convexity of $C_n(q)$ over the real line.

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Theorem 1.1. The q-Catalan polynomials $C_n(q)$ are strictly convex over $\mathbb{R}$ for every integer $n \geq 2$.

We like to remark that, by contrast, the closely related central q-binomial coefficients $\begin{bmatrix} 2n \\ n \end{bmatrix}_q$ are not generally convex, since their degree is not always even. They do seem to be convex in the case that n is even, and it might be possible to prove this along the same lines as Theorem 1.1.

We now describe the organization of our paper. Section 2 covers basic properties of the q-Catalan polynomials and their derivatives, culminating in Corollary 2.5, which shows that it suffices to prove convexity for $q \in (-1, 0)$, provided that an additional technical condition holds. This is achieved in Proposition 3.2. Two other lemmas in Section 3, namely Lemma 3.5 and Lemma 3.6, provide the inductive step for an induction proof of convexity for $q \in (-1, 0)$ (see Theorem 4.1). Our central result, Theorem 1.1, then follows by putting everything together. The final section, Section 6, adds a convexity result for the generating function of the integer partitions and concludes with a question for the inspired reader.

2. Preliminary results

In this section, we shall list some basic properties of the MacMahon q-Catalan polynomials and a few other results pertinent to our principal goal. Throughout this paper $f'(q)$ means derivative with respect to the variable q, i.e., $f'(q) = \frac{d}{dq} f(q)$.

Proposition 2.1. The following statements hold.

(a) $C_n(0) = 1, C_n'(0) = 0$, and $C_n''(0) = 2$ (the latter for $n \geq 2$).

(b) $C_n(1) = C_n = \frac{1}{n+1}\binom{2n}{n}$, and $C_n'(1) = \binom{n}{2} C_n$.

(c) $C_n(-1) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$, while $C_n'(-1) = -\binom{n}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor}$ and

$$C_n''(-1) = \frac{1}{12}\binom{n}{\lfloor \frac{n}{2} \rfloor} \cdot \begin{cases} n^2(n+1)(3n-5) & \text{if } n \text{ is even;} \\ (n^2-1)(3n^2-2n-2) & \text{if } n \text{ is odd.} \end{cases}$$

(d) $C_n(q) > 0$ for any real number q and any $n \in \mathbb{N}$.

(e) $C_n(q)$ is strictly increasing and strictly convex whenever $q > 0$.

Proof. (a)–(c): We may express $C_n(q) = f_n(q) C_{n-1}(q)$ where

$$f_n(q) = \frac{(1+q^n)(1-q^{2n-1})}{1-q^{n+1}}. \tag{2.1}$$

Then, the assertions become rather elementary based on the product rule $C_n' = f_n' C_{n-1} + f_n C_{n-1}', C_n'' = f_n'' C_{n-1} + 2 f_n' C_{n-1}' + f_n C_{n-1}''$ and induction on n.

(d)–(e): The *major index* of a Dyck path D, denoted $\mathrm{maj}(D)$, is the sum over all i for which (i, j) is a *valley* of D. The MacMahon's q-Catalan [6] is a generating function for the major index:

$$\sum_{D\in\mathcal{D}_n} q^{\mathrm{maj}(D)} = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q.$$

It becomes immediate, from this combinatorial fact, that $C_n(q)$ is indeed a polynomial and each coefficient of $C_n(q)$ is non-negative. Thus, $C_n'(q) > 0$ and $C_n''(q) > 0$ for $q > 0$. We also would like to remark that $C_n(q)$ has degree $n(n-1)$ and the only vanishing coefficients are those of q and $q^{n(n-1)-1}$. By its very definition, each factor $[j] = \frac{1-q^j}{1-q}$ of $C_n(q)$ is a product of cyclotomic polynomials, hence the roots lie on the unit circle. In particular, any possible real root can only be $q = \pm 1$. From parts (b) and (c) above, it is clear that $C_n(\pm 1) \neq 0$. Therefore, $C_n(q) \neq 0$ for any $q \in \mathbb{R}$. On the other hand, $C_n(q)$ is continuous (as a polynomial) and hence by the intermediate value theorem it must be either always negative or always positive-valued. Computing at any real q, say $C_n(0) = 1$, decidedly proves $C_n(q) > 0$ for all $q \in \mathbb{R}$. □

Rewrite $C_n(q) = \prod_{j=2}^n \frac{1-q^{n+j}}{1-q^j}$ so that $C_n'(q) = C_n(q)Q_n(q)$, where we have introduced the rational function

$$Q_n(q) := \sum_{j=2}^{n} \left(\frac{1-q^{n+j}}{1-q^j}\right)^{-1}\left(\frac{1-q^{n+j}}{1-q^j}\right)'.$$

A routine calculation also shows that

$$Q_n(q) = \sum_{j=2}^{n} \frac{jq^{j-1}}{1-q^j} - \sum_{j=n+2}^{2n} \frac{jq^{j-1}}{1-q^j}.$$

Note 2.2. Although $Q_n(q)$ appears to have singularities when $q = \pm 1$, it does *not!* The reason is that, by Proposition 2.1(d), the polynomial $C_n(q)$ never equals zero for any real q. Thus the rational function $Q_n(q) = \frac{C_n'(q)}{C_n(q)}$ is well-defined and analytic for all $q \in \mathbb{R}$. In short, $Q_n(q)$ has finite values at $q = \pm 1$ (at least in the limit) and it is a smooth function over $\mathbb{R}$. In particular, Proposition 2.1(c) implies that $Q_n(1) = \binom{n}{2}$ and $Q_n(-1) = -\binom{n}{2}$ are indeed finite. Moreover, it follows that all derivatives are also finite and bounded over the compact interval $|q| \leq 1$.

Lemma 2.3. Denote $N := n(n-1)$. Then, the following identity holds:

$$qQ_n(q) + q^{-1}Q_n(q^{-1}) = N.$$

Proof. Simply note that $\frac{q^j}{1-q^j} + \frac{(q^{-1})^j}{1-(q^{-1})^j} = -1$, and take the sum over j after multiplying by j. □

Lemma 2.4. The following relation holds:

$$q^{N-2}C_n''(q^{-1}) = q^2C_n''(q) + (N-1)\cdot[NC_n(q) - 2qC_n'(q)].$$

Proof. Compute two derivatives in $C_n(q) = q^N C_n(q^{-1})$ by taking advantage of $C_n'(q) = Nq^{-1}C_n(q) - q^{N-2}C_n'(q^{-1})$ (see Lemma 2.3) and repeat this in the reverse form $q^{N-2}C_n'(q^{-1}) = Nq^{-1}C_n(q) - C_n'(q)$. We have

$$\begin{aligned} C_n''(q) &= -Nq^{-2}C_n(q) + Nq^{-1}C_n'(q) - (N-2)q^{N-3}C_n'(q^{-1}) + q^{N-4}C_n''(q^{-1}) \\ &= -N(N-1)q^{-2}C_n(q) + 2(N-1)q^{-1}C_n'(q) + q^{N-4}C_n''(q^{-1}) \\ &= (N-1)q^{-1}\cdot[2C_n'(q) - Nq^{-1}C_n(q)] + q^{N-4}C_n''(q^{-1}). \end{aligned}$$

The claim follows upon multiplying through by q^2 and swapping terms. $\square$

Corollary 2.5. If, for $-1 < q < 0$ and $n \geq 2$, the polynomial $C_n(q)$ is strictly convex and $F_n(q) := \binom{n}{2} - qQ_n(q) \geq 0$, then $C_n(q)$ is also strictly convex for $q < -1$.

Proof. Observe that

$$NC_n(q) - 2qC_n'(q) = NC_n(q) - 2qC_n(q)Q_n(q) = 2C_n(q)\left[\binom{n}{2} - qQ_n(q)\right],$$

so Lemma 2.4 reads

$$q^{N-2}C_n''(q^{-1}) = q^2C_n''(q) + 2(N-1)C_n(q)F_n(q). \tag{2.2}$$

We know $C_n(q)$ is always positive-valued. The factors q^{N-2} and q^2 are both positive, therefore if $C_n''(q) > 0$ and $F_n(q) \geq 0$ then equation (2.2) implies $C_n''(q^{-1}) > 0$. The conclusion follows because $q^{-1} < -1$ if $-1 < q < 0$. $\square$

3. Auxiliary inequalities

In the present section, we first aim to prove Proposition 3.2 which accounts for one of the hypotheses of Corollary 2.5. Let us commence with some intermediate results.

Lemma 3.1. If $0 < q < 1$ and $n \geq 2$, then $Q_n'(q)$ is positive.

Proof. We use induction on $n \geq 2$. Clearly $Q_2'(q) = \frac{2(1-q^2)}{(1+q^2)^2} > 0$. Assume the claim holds for n. Let us denote $R_n(q) := q^2Q_{n+1}'(q) - q^2Q_n'(q)$ and further split $R_n(q) = R_n^{(1)}(q) + R_n^{(2)}(q)$ where

$$\begin{aligned} R_n^{(1)}(q) &:= \frac{(n+2)q^{n+2}(q^{n+2}+n+1)}{(1-q^{n+2})^2} - \frac{(2n+1)q^{2n+1}(q^{2n+1}+2n)}{(1-q^{2n+1})^2}, \\ R_n^{(2)}(q) &:= \frac{(n+1)q^{n+1}(q^{n+1}+n)}{(1-q^{n+1})^2} - \frac{(2n+2)q^{2n+2}(q^{2n+2}+2n+1)}{(1-q^{2n+2})^2}. \end{aligned}$$

These expressions motivate defining the function

$$f_-(t,x) := \frac{tx^t(t-1+x^t)}{(1-x^t)^2}$$

whose derivative takes the form

$$\frac{\partial}{\partial t} f_-(t,x) = ta_-(x^t) - b_-(x^t),$$

where

$$a_-(y) := \frac{2y}{(1-y)^2} + \frac{y(1+y)\log y}{(1-y)^3} \quad \text{and} \quad b_-(y) := \frac{y}{1-y} + \frac{y\log y}{(1-y)^2}.$$

By the mean value theorem there is a $\tau \in (n+2, 2n+1)$ satisfying

$$R_n^{(1)}(q) = f_-(n+2,x) - f_-(2n+1,x) = (n-1)[-\tau a_-(x^\tau) + b_-(x^\tau)].$$

Let us check that $-a_-(y) = -\frac{y(1+y)}{(1-y)^3}\left[\frac{2(1-y)}{1+y} + \log y\right] > 0$ for $y \in (0,1)$. If $h(y) := \frac{2(1-y)}{1+y} + \log y$ then $h(1) = 0$ and $\frac{dh(y)}{dy} = \frac{(1-y)^2}{y(1+y)^2} > 0$ whenever $y > 0$, which shows $h(y) > 0$ for $y > 1$ and $h(y) < 0$ for $y < 1$. The implication on $-a_-(y) > 0$ is clear. It follows that $-\tau a_-(x^\tau) \geq -4a_-(x^\tau)$, since $\tau > n+2 \geq 4$. So in order to obtain $R_n^{(1)}(q) > 0$, it suffices to prove

$$-4a_-(y) + b_-(y) = \frac{y(y^2+6y-7-(5y+3)\log y)}{(1-y)^3} > 0.$$

This is equivalent to $\frac{y^2+6y-7}{5y+3} - \log y > 0$, which can for example be shown by noting that equality holds when $y = 1$, while the derivative of the left side is $\frac{(1-y)^2(5y-9)}{y(5y+3)^2} < 0$.

Thus $R_n^{(1)}(q) > 0$. An analogous argument works for $R_n^{(2)}(q) > 0$. However, we opt to display a more elementary method, namely that

$$R_n^{(2)}(q) = f_-(n+1,q) - f_-(2n+2,q) = \frac{(n+1)q^{n+1}(n-q^{n+1})}{(1+q^{n+1})^2} > 0.$$

So $R_n(q) > 0$, and based on the induction hypothesis, we can infer that $q^2Q'_{n+1}(q) = q^2Q'_n(q) + R_n(q) > 0$. This completes the proof that $Q'_n(q)$ is indeed positive, as required. □

Proposition 3.2. If $-1 < q < 1$ then $F_n(q) = \binom{n}{2} - qQ_n(q)$ is positive with $F_n(\pm 1) = 0$ and $F_n(0) = \binom{n}{2}$.

Proof. Since $qC'_n(q) = C_n(q) \cdot qQ_n(q) = C_n(q)\left[\binom{n}{2} - F_n(q)\right]$ and $C_n(q) \neq 0$, we notice that $F_n(q) = \binom{n}{2} - \frac{qC'_n(q)}{C_n(q)}$ is well-defined for any $q \in \mathbb{R}$. Obviously $F_n(0) = \binom{n}{2}$. Applying Proposition 2.1 (b) and (c) verifies $F_n(\pm 1) = 0$.

Consider first the case that $0 < q < 1$: by Proposition 2.1, $C_n(q) > 0$ and $C_n'(q) = C_n(q)Q_n(q) > 0$ imply $Q_n(q) > 0$; Lemma 3.1 gives $Q_n'(q) > 0$. Combining, we get $F_n'(q) = -Q_n(q) - qQ_n'(q) < 0$, i.e., $F_n(q)$ is decreasing. Thus $F_n(q) > F_n(1) = 0$, and we are done.

If $-1 < q < 0$, change variables to $q = -t$. We will show that $F_n(-t) \geq F_n(t) > 0$ for $0 < t < 1$, completing the proof. To this end, break up the sums according to parity to get

$$F_n(-t) = \binom{n}{2} - \sum_{\substack{2\leq j\leq n \\ j \text{ even}}} \frac{jt^j}{1-t^j} + \sum_{\substack{2\leq j\leq n \\ j \text{ odd}}} \frac{jt^j}{1+t^j} + \sum_{\substack{n+2\leq j\leq 2n \\ j \text{ even}}} \frac{jt^j}{1-t^j} - \sum_{\substack{n+2\leq j\leq 2n \\ j \text{ odd}}} \frac{jt^j}{1+t^j}$$

$$F_n(t) = \binom{n}{2} - \sum_{\substack{2\leq j\leq n \\ j \text{ even}}} \frac{jt^j}{1-t^j} - \sum_{\substack{2\leq j\leq n \\ j \text{ odd}}} \frac{jt^j}{1-t^j} + \sum_{\substack{n+2\leq j\leq 2n \\ j \text{ even}}} \frac{jt^j}{1-t^j} + \sum_{\substack{n+2\leq j\leq 2n \\ j \text{ odd}}} \frac{jt^j}{1-t^j}.$$

So it is enough to justify that

$$\sum_{\substack{2\leq j\leq n \\ j \text{ odd}}} \frac{jt^j}{1+t^j} - \sum_{\substack{n+2\leq j\leq 2n \\ j \text{ odd}}} \frac{jt^j}{1+t^j} \geq - \sum_{\substack{2\leq j\leq n \\ j \text{ odd}}} \frac{jt^j}{1-t^j} + \sum_{\substack{n+2\leq j\leq 2n \\ j \text{ odd}}} \frac{jt^j}{1-t^j},$$

or equivalently,

$$\sum_{\substack{2\leq j\leq n \\ j \text{ odd}}} \frac{2jt^j}{1-t^{2j}} \geq \sum_{\substack{n+2\leq j\leq 2n \\ j \text{ odd}}} \frac{2jt^j}{1-t^{2j}} \quad \Longleftrightarrow \quad \sum_{\substack{2\leq j\leq n \\ j \text{ odd}}} \frac{2j}{t^{-j}-t^j} \geq \sum_{\substack{n+2\leq j\leq 2n \\ j \text{ odd}}} \frac{2j}{t^{-j}-t^j}.$$

In fact, this inequality holds (strictly) term-by-term: depending on the parity of n,

$$\frac{2j}{t^{-j}-t^j} > \frac{2(n+j)}{t^{-(n+j)}-t^{n+j}} \qquad \text{or} \qquad \frac{2j}{t^{-j}-t^j} > \frac{2(n+j-1)}{t^{-(n+j-1)}-t^{n+j-1}}. \tag{3.1}$$

The former applies for n even, the latter for n odd. Choose $\theta > 0$ such that $t^{-1} = e^\theta$ and define the auxiliary function $g(x) := \frac{\sinh(\theta x)}{x} = \frac{t^{-x}-t^x}{2x}$. Now, inequality (3.1) amounts to $g(n+j) > g(j)$ or $g(n+j-1) > g(j)$. This *monotonicity*, however, follows from the elementary observation

$$g'(x) = \frac{\theta x \cosh(\theta x) - \sinh(\theta x)}{x^2} > 0$$

(equivalent to $\tanh(\theta x) < \theta x$, which is well known and follows from the mean value theorem). We conclude $F_n(-t) \geq F_n(t) > 0$ for any $0 < t < 1$. The proof is complete. □

Note 3.3. Figure 1 illustrates positivity and concavity (the latter is left to the interested reader to check) of $F_4(q)$ in the range $-1 < q < 1$.

Corollary 3.4. For $q < -1$, $C_n(q)$ is strictly decreasing.

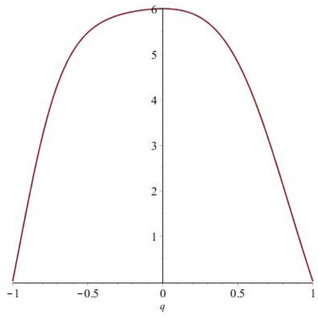

FIGURE 1. Positivity and concavity of $F_4(q)$.

Proof. By Proposition 3.2, we have $q^{-1}Q_n(q^{-1}) < \binom{n}{2}$, and hence $qQ_n(q) > \binom{n}{2} > 0$ by Lemma 2.3; this means that $Q_n(q) < 0$. Therefore $C_n'(q) = C_n(q)Q_n(q) < 0$ holds since $C_n(q)$ is always positive by Proposition 2.1(d). $\square$

For the following two lemmas, which provide the induction step in proving Theorem 4.1 later, we designate

$$\varphi_j(q) := \frac{jq^j[j-1+q^j]}{(1-q^j)^2} = -\frac{j(j-1)}{1-q^{-j}} + \frac{j^2}{(1-q^{-j})^2}.$$

Note that this is equal to the function $f_-(j,q)$ used earlier, but since we are not distinguishing the two parity cases any longer in the formulation of the next results, we use this simpler notation.

Lemma 3.5. If $n = 2m$ is an even positive integer and $-1 < q < 0$ a real number, then

$$K_n(q) := \varphi_n(q) - \varphi_{2n}(q) + \varphi_{n+1}(q) - \varphi_{2n-1}(q) > 0.$$

Proof. Observe that $\varphi_n(q) - \varphi_{2n}(q) = \frac{nq^n(n-1-q^n)}{(1+q^n)^2} = \frac{n(n-1)}{1+q^{-n}} - \frac{n^2}{(1+q^{-n})^2}$ after a direct simplification. The substitution $q = -x$ (so that $0 < x < 1$) and notation $\hat{K}_{2m}(x) := K_{2m}(-x)$ give away $\varphi_{2m}(-x) - \varphi_{4m}(-x) = \frac{2m(2m-1)}{1+(-x)^{-2m}} - \frac{4m^2}{(1+(-x)^{-2m})^2} = \frac{2m(2m-1)}{1+x^{-2m}} - \frac{4m^2}{(1+x^{-2m})^2}$. On the other hand, from the definition $\varphi_{2m+1}(-x) = -\frac{2m(2m+1)}{1-(-x)^{-(2m+1)}} + \frac{(2m+1)^2}{(1-(-x)^{-(2m+1)})^2}$ and $\varphi_{4m-1}(-x) = -\frac{(4m-1)(4m-2)}{1-(-x)^{-(4m-1)}} + \frac{(4m-1)^2}{(1-(-x)^{-(4m-1)})^2}$. Armed with these properties, we proceed to rearrange the terms of $\hat{K}_{2m}(x)$

in the following way:

$$\begin{aligned}
\hat{K}_{2m}(x) &= \frac{2m(2m-1)}{1+x^{-2m}} - \frac{4m^2}{(1+x^{-2m})^2} - \frac{2m(2m+1)}{1+x^{-(2m+1)}} + \frac{(2m+1)^2}{(1+x^{-(2m+1)})^2} \\
&\quad + \frac{(4m-1)(4m-2)}{1+x^{-(4m-1)}} - \frac{(4m-1)^2}{(1+x^{-(4m-1)})^2} \\
&= \left(\frac{(2m+1)^2}{(1+x^{-(2m+1)})^2} - \frac{(2m+1)^2}{(1+x^{-(4m-1)})^2}\right) + \left(\frac{2m(2m-1)}{1+x^{-(4m-1)}} - \frac{4m^2}{(1+x^{-2m})^2}\right) \\
&\quad + \left(\frac{6m(m-1)}{1+x^{-(4m-1)}} - \frac{12m(m-1)}{(1+x^{-(4m-1)})^2}\right) \\
&\quad + \left(\frac{2m(2m-1)}{1+x^{-2m}} - \frac{2m(2m+1)}{1+x^{-(2m+1)}} + \frac{6m^2-4m+2}{1+x^{-(4m-1)}}\right).
\end{aligned}$$

Next, we treat each of the terms individually. Putting the following four inequalities together proves that $\hat{K}_{2m}(x) > 0$ and hence $K_{2m}(q) > 0$.

Inequality 1: Clearly,

$$\frac{(2m+1)^2}{(1+x^{-(2m+1)})^2} - \frac{(2m+1)^2}{(1+x^{-(4m-1)})^2} \geq 0$$

for all $m \geq 1$, since $4m-1 \geq 2m+1$.

Inequality 2: Since $x^{2m} - x^{4m} > 0$, we have

$$\begin{aligned}
\frac{2m(2m-1)}{1+x^{-(4m-1)}} - \frac{4m^2}{(1+x^{-2m})^2} &= \frac{2mx^{4m-1}(-x^{4m} + (4m-2)x^{2m} - 2mx + 2m - 1)}{(1+x^{2m})^2(x^{4m-1}+1)} \\
&> \frac{2mx^{4m-1}((4m-3)x^{2m} - 2mx + 2m - 1)}{(1+x^{2m})^2(x^{4m-1}+1)}.
\end{aligned}$$

Now just note that the minimum of the factor $(4m-3)x^{2m} - 2mx + 2m - 1$ is attained at $x = (4m-3)^{-1/(2m-1)}$ and equal to $(2m-1)(1-(4m-3)^{-1/(2m-1)}) \geq 0$ (as can be shown by elementary calculus) to prove that this term is positive.

Inequality 3: Because $x^{-1} > 1$, we have

$$\frac{6m(m-1)}{1+x^{-(4m-1)}} - \frac{12m(m-1)}{(1+x^{-(4m-1)})^2} = \frac{6m(m-1)(x^{-(4m-1)}-1)}{(1+x^{-(4m-1)})^2} \geq 0.$$

Inequality 4: We want to show that

$$\frac{2m(2m-1)}{1+x^{-2m}} - \frac{2m(2m+1)}{1+x^{-(2m+1)}} + \frac{6m^2-4m+2}{1+x^{-(4m-1)}} > 0.$$

For $m = 1$, this readily reduces to $\frac{2}{1+x^{-2}} - \frac{2}{1+x^{-3}} > 0$, which is clearly true. So, assume that $m \geq 2$. Next, observe that $6m^2 - 4m + 2 \geq 2m^2 + 5m$ for all $m \geq 2$ (this seemingly arbitrary estimate will simplify expressions later). So it suffices to prove that

$$A = \frac{2m(2m-1)}{1+x^{-2m}} - \frac{2m(2m+1)}{1+x^{-(2m+1)}} + \frac{2m^2+5m}{1+x^{-(4m-1)}} > 0.$$

To this end, note first that

$$\begin{aligned}
\tfrac{1}{mx^{2m}}&(1+x^{2m})(1+x^{2m+1})(1+x^{4m-1})A \\
&= (2m+1)x^{6m} + (6m+3-(2m-3)x)x^{4m-1} \\
&\quad + (2m+5-4x^2)x^{2m-1} - 2(2m+1)x + 2(2m-1) \\
&> (2m+1)x^{2m-1} - 2(2m+1)x + 2(2m-1).
\end{aligned}$$

This final expression has its minimum at $x = (\frac{2}{2m-1})^{1/(2m-2)}$, and the minimum is equal to $2(2m-1) - \frac{4(m-1)(2m+1)}{2m-1}(\frac{2}{2m-1})^{1/(2m-2)}$. We only have to show that this is positive, which is equivalent to

$$\left(\tfrac{2(m-1)(2m+1)}{(2m-1)^2}\right)^{2m-2} < \tfrac{2m-1}{2} \tag{3.2}$$

after some straightforward manipulations. Since

$$\left(\tfrac{2(m-1)(2m+1)}{(2m-1)^2}\right)^{2m-2} = \left(1 + \tfrac{1}{2m-1} - \tfrac{2}{(2m-1)^2}\right)^{2m-2} < \left(1 + \tfrac{1}{2m-1}\right)^{2m-1} < e$$

(it is well known that $(1+1/n)^n$ is increasing and converges to e), the claim materializes soon as $\frac{2m-1}{2} > e$, i.e., $m \geq 4$. For $m = 2$ and $m = 3$, one can verify directly that (3.2) holds. This completes the proof of the fourth and final inequality. □

Lemma 3.5 has the following counterpart for odd integers, which we prove by means of a different approach.

Lemma 3.6. If $n = 2m+1 > 1$ is an odd positive integer and $-1 < q < 0$ is a real number, then

$$L_n(q) := -\varphi_{2n-1} - \varphi_{2n-3} + (\varphi_{n-1} - \varphi_{2n-2}) + \varphi_{n+1} + (\varphi_n - \varphi_{2n}) + \varphi_n > 0.$$

Proof. Make the change of variables $x := -q \in (0,1)$ as in the previous lemma. Let us first consider the case that $m = 1$ ($n = 3$): here,

$$L_3(q) = \frac{x^2(1-x)^2 N}{(1-x^3)^2(1+x^5)^2},$$

where the final factor in the numerator is

$$\begin{aligned}
N &= 2 - 2x + 18x^3 + 43x^4 + 60x^5 + 41x^6 + 14x^7 \\
&\qquad - 2x^8 - 12x^9 - 12x^{10} - 12x^{11} - 6x^{12} \\
&\geq 2 - 2x + (18+43+60+41+14)x^7 - (2+12+12+12+6)x^8 \\
&= 2(1-x) + 44x^7(4-x) > 0.
\end{aligned}$$

From now on, we can assume that $m > 1$. Next, we set

$$f_-(t,x) := \frac{tx^t(t-1+x^t)}{(1-x^t)^2} \quad \text{and} \quad f_+(t,x) := \frac{tx^t(t-1-x^t)}{(1+x^t)^2},$$

cf. the proof of Lemma 3.1. Observe that

$$\varphi_k = \begin{cases} \frac{kx^k(k-1+x^k)}{(1-x^j)^2} = f_-(k,x) & k \text{ even}, \\ -\frac{kx^k(k-1-x^k)}{(1+x^k)^2} = -f_+(k,x) & k \text{ odd}, \end{cases}$$

as well as

$$\varphi_k - \varphi_{2k} = \begin{cases} \frac{kx^k(k-1-x^k)}{(1+x^k)^2} = f_+(k,x) & k \text{ even}, \\ -\frac{kx^k(k-1+x^k)}{(1-x^k)^2} = -f_-(k,x) & k \text{ odd}. \end{cases}$$

Hence we have

$$\begin{aligned} L_n(q) = {} & f_+(4m+1,x) + f_+(4m-1,x) + (f_+(2m,x) - f_+(2m+1,x)) \\ & + (f_-(2m+2,x) - f_-(2m+1,x)). \end{aligned} \tag{3.3}$$

We distinguish two different cases, one for "large" x, and one for "small" x.

Case 1: $x^{2m+2} \geq \frac{1}{10}$**.** We apply the mean value theorem to the differences $f_+(2m,x) - f_+(2m+1,x)$ and $f_-(2m+2,x) - f_-(2m+1,x)$. Recall that $\frac{\partial}{\partial t} f_-(t,x) = ta_-(x^t) - b_-(x^t)$, where

$$a_-(y) := \frac{2y}{(1-y)^2} + \frac{y(1+y)\log y}{(1-y)^3} \quad \text{and} \quad b_-(y) := \frac{y}{1-y} + \frac{y\log y}{(1-y)^2}.$$

Likewise, we have $\frac{\partial}{\partial t} f_+(t,x) = ta_+(x^t) - b_+(x^t)$, where

$$a_+(y) := \frac{2y}{(1+y)^2} + \frac{y(1-y)\log y}{(1+y)^3} \quad \text{and} \quad b_+(y) := \frac{y}{1+y} + \frac{y\log y}{(1+y)^2}.$$

It follows that $f_+(2m,x) - f_+(2m+1,x) = -\tau_1 a_+(x^{\tau_1}) + b_+(x^{\tau_1})$ for some $\tau_1 \in (2m, 2m+1)$. Likewise, $f_-(2m+2,x) - f_-(2m+1,x) = \tau_2 a_-(x^{\tau_2}) - b_-(x^{\tau_2})$ for some $\tau_2 \in (2m+1, 2m+2)$. It is routine to verify that

- $a_+(y)$ is positive and increasing for $\frac{1}{10} \leq y \leq 1$,
- $b_+(y)$ is increasing for $\frac{1}{10} \leq y \leq 1$,
- $a_-(y)$ is negative and decreasing for $0 < y < 1$,
- $b_-(y)$ is decreasing for $0 < y < 1$,
- $a_-(y)$ and $b_-(y)$ are bounded at 1, in spite of the factor $1-y$ in the denominator, with $\lim_{y\to 1^-} a_-(y) = -\frac{1}{6}$ and $\lim_{y \to 1^-} b_-(y) = -\frac{1}{2}$.

We are assuming that $x^{2m+2} \geq \frac{1}{10}$, so $x^{2m} \geq x^{\tau_1} \geq x^{2m+1} \geq x^{\tau_2} \geq x^{2m+2} \geq \frac{1}{10}$. By monotonicity of a_+, a_-, b_+ and b_-, we have

$$\begin{aligned}
&(f_+(2m,x) - f_+(2m+1,x)) + (f_-(2m+2,x) - f_-(2m+1,x)) \\
&\qquad = -\tau_1 a_+(x^{\tau_1}) + b_+(x^{\tau_1}) + \tau_2 a_-(x^{\tau_2}) - b_-(x^{\tau_2}) \\
&\qquad = -(\tau_1+1)a_+(x^{\tau_1}) + \tau_2 a_-(x^{\tau_2}) + \big(a_+(x^{\tau_1}) + b_+(x^{\tau_1}) - b_-(x^{\tau_2})\big) \\
&\qquad \geq -(\tau_1+1)a_+(x^{2m}) + \tau_2 a_-(x^{2m}) + \big(a_+(\tfrac{1}{10}) + b_+(\tfrac{1}{10}) - b_-(\tfrac{1}{10})\big) \\
&\qquad > -(2m+2)(a_+(x^{2m}) - a_-(x^{2m}))
\end{aligned}$$

as $a_+(\frac{1}{10}) + b_+(\frac{1}{10}) - b_-(\frac{1}{10}) \approx 0.083 > 0$.

On the other hand,

$$\begin{aligned}
&f_+(4m+1,x) + f_+(4m-1,x) \\
&\quad = \frac{(4m+1)x^{4m+1}(4m - x^{4m-1})}{(1+x^{4m+1})^2} + \frac{(4m-1)x^{4m-1}(4m-2-x^{4m-1})}{(1+x^{4m-1})^2} \\
&\quad \geq \frac{(4m+1)x^{4m+1}(4m-1)}{(1+x^{4m})^2} + \frac{(4m-1)x^{4m}(4m-3)}{(1+x^{4m})^2} \\
&\quad = \frac{(4m-1)x^{4m}(4m-3+(4m+1)x)}{(1+x^{4m})^2} \\
&\quad \geq \frac{(4m-1)x^{4m}(4m-3+(4m+1)10^{-1/(2m+2)})}{(1+x^{4m})^2}
\end{aligned}$$

where we have used the monotonicity of the function $y \mapsto \frac{y}{(1+y)^2}$ to deduce that $\frac{x^{4m-1}}{(1+x^{4m-1})^2} \geq \frac{x^{4m}}{(1+x^{4m})^2}$. In order to obtain the inequality $L_n(q) > 0$, it is therefore sufficient that

$$\frac{(4m-1)x^{4m}(4m-3+(4m+1)10^{-1/(2m+2)})}{(1+x^{4m})^2} \geq (2m+2)(a_+(x^{2m}) - a_-(x^{2m})),$$

which is equivalent to

$$\frac{(4m-1)(4m-3+(4m+1)10^{-1/(2m+2)})}{2m+2} \geq h(x^{2m}),$$

where

$$h(y) = \frac{(a_+(y) - a_-(y))(1+y^2)^2}{y^2} = \frac{8(1+y^2)^2(1-y^2+(1+y^2)\log y)}{(y^2-1)^3}.$$

This auxiliary function is decreasing for $0 < y < 1$, which can for example be seen by substituting $y^2 = 1 - t$ and noting that the resulting function

$$h(\sqrt{1-t}) = \frac{8}{3} + \sum_{j=2}^{\infty} \frac{4(j-1)(j^2+j+6)}{j(j+1)(j+2)(j+3)} t^j$$

has only positive Taylor coefficients. So $11.23 \approx h(\frac{1}{10}) \geq h(x^{2m})$. On the other hand, $\frac{(4m-1)(4m-3+(4m+1)10^{-1/(2m+2)})}{2m+2} > 12$ for $m \geq 2$, completing the proof in this case.

Case 2: $x^{2m+2} \leq \frac{1}{10}$**.** In this case, we use the inequalities

$$\begin{aligned} u_+(k,x) &:= kx^k(k-1-x^k)(1-x^k) \geq f_+(k,x) = \frac{kx^k(k-1-x^k)}{(1+x^k)^2} \\ &\geq kx^k(k-1-x^k)(1-2x^k) =: \ell_+(k,x). \end{aligned}$$

The first inequality holds for all $k \geq 1$ and $x^k \leq \frac{\sqrt{5}-1}{2}$, since $(1-t)(1+t)^2 \geq 1$ for $t \leq \frac{\sqrt{5}-1}{2}$. The second inequality is in fact valid for all $x \in (0,1)$, since $(1-2t)(1+t)^2 \leq 1$ is true for all $t > 0$. Similarly,

$$\begin{aligned} u_-(k,x) &:= kx^k(k-1+x^k)(1+3x^k) \geq -f_-(k,x) = \frac{kx^k(k-1+x^k)}{(1-x^k)^2} \\ &\geq kx^k(k-1+x^k)(1+2x^k) := \ell_-(k,x), \end{aligned}$$

which holds for all $k \geq 1$ and $x^k \leq \frac{5-\sqrt{13}}{6}$. Once more, the second inequality holds for all $x \in (0,1)$.

We apply these inequalities to each of the terms of $L_n(q)$ in (3.3). The upper bounds apply since $x^{2m+1} \leq (\frac{1}{10})^{(2m+1)/(2m+2)} \leq 10^{-5/6} < \frac{5-\sqrt{13}}{6} < \frac{\sqrt{5}-1}{2}$. Hence we obtain

$$\begin{aligned} L_n(q) \geq\ & \ell_+(4m+1,x) + \ell_+(4m-1,x) + \ell_+(2m,x) \\ & - u_+(2m+1,x) + \ell_-(2m+2,x) - u_-(2m+1,x). \end{aligned}$$

Now expand all these expressions and collect the terms involving m^2 and m respectively. This yields

$$\begin{aligned} L_n(q) \geq\ & m^2\Big(4(1-x)^2x^{2m} + 8(2-x+2x^2-x^3+x^5)x^{4m-1} - 32(1+x^4)x^{8m-2}\Big) \\ & + m\Big(2(x-1)(1+3x)x^{2m} - 2(6-x-2x^2+2x^3-7x^5)x^{4m-1} \\ & + 4(1-x^3)^2x^{6m} + (20-12x^4)x^{8m-2} + 8(1+x^6)x^{12m-3}\Big) \\ & + \Big(2x^{2m+2} + 2x^{4m-1} + 6x^{4m+4} - 4x^{6m+3} + 4x^{6m+6} - 3x^{8m-2} - x^{8m+2} \\ & - 2x^{12m-3} + 2x^{12m+3}\Big) \\ =:\ & c_2(m,x)m^2 + c_1(m,x)m + c_0(m,x). \end{aligned}$$

Let us estimate the three coefficients $c_0(m,x)$, $c_1(m,x)$ and $c_2(m,x)$. First, since $4m-1, 4m+4 \leq 6m+3, 8m-2, 8m+2$ and $6m+6 \leq 12m-3$, it is clear that we can drop all terms from $c_0(m,x)$ except for the first:

$$c_0(m,x) > 2x^{2m+2}.$$

Next, dropping positive terms from $c_1(m,x)$ yields

$$c_1(m,x) > 2(x-1)(1+3x)x^{2m} - 12x^{4m-1}.$$

Finally, since $4m-2 \geq 2m+2$, we have $x^{4m-2} \leq x^{2m+2} \leq \frac{1}{10}$ and thus $32(1+x^4)x^{8m-2} \leq \frac{32}{10}(1+x^4)x^{4m}$, which implies that

$$\begin{aligned} c_2(m,x) &\geq 4(1-x)^2x^{2m} + \left(16 - \tfrac{56x}{5} + 16x^2 - 8x^3 + \tfrac{24x^5}{5}\right)x^{4m-1} \\ &> 4(1-x)^2x^{2m} + (16 - 8\sqrt{2}x + 16x^2 - 8x^3)x^{4m-1} \\ &= 4(1-x)^2x^{2m} + (12 + (2-2\sqrt{2}x)^2 + 8x^2(1-x))x^{4m-1} \\ &\geq 4(1-x)^2x^{2m} + 12x^{4m-1}. \end{aligned}$$

Combining these inequalities, we arrive at

$$L_n(q) > x^{2m}\left(4(1-x)^2m^2 + 2(x-1)(1+3x)m + 2x^2\right) + 12(m^2-m)x^{4m-1}.$$

Thus it remains to show that

$$4(1-x)^2m^2 + 2(x-1)(1+3x)m + 2x^2 + 12(m^2-m)x^{2m-1} \geq 0.$$

The quadratic polynomial $4(1-x)^2m^2 + 2(x-1)(1+3x)m + 2x^2$ reaches minimum $-\frac{2m}{m+1}$ (attained at $x = \frac{m}{m+1}$), and it is positive outside of the interval $[\frac{2m^2+m-\sqrt{m+2m^2}}{1+3m+2m^2}, \frac{2m^2+m+\sqrt{m+2m^2}}{1+3m+2m^2}]$. Hence we may assume that $x \geq \frac{2m^2+m-\sqrt{m+2m^2}}{1+3m+2m^2}$. But then it follows that

$$\begin{aligned} 12(m^2-m)x^{2m-1} &\geq 12(m^2-m)\left(\tfrac{2m^2+m-\sqrt{m+2m^2}}{1+3m+2m^2}\right)^{2m-1} \\ &= 12(m^2-m)\left(1 - \tfrac{1}{2m+1-\sqrt{2m^2+m}}\right)^{2m-1} \\ &\geq 12(m^2-m)\left(1 - \tfrac{1}{(2-\sqrt{2})m}\right)^{2m}. \end{aligned}$$

The final expression is increasing in m (the well-known fact that $(1-\frac{a}{t})^t$ is increasing for $t > a$ was already mentioned earlier) and approximately equal to 3.68 for $m = 5$. For $m \in \{2,3,4\}$, one can verify directly that $12(m^2-m)(\frac{2m^2+m-\sqrt{m+2m^2}}{1+3m+2m^2})^{2m-1} > 2$. So in all cases, we have

$$4(1-x)^2m^2 + 2(x-1)(1+3x)m + 2x^2 + 12(m^2-m)x^{2m-1} > -\frac{2m}{m+1} + 2 > 0,$$

completing the proof. □

4. Proof of the main result

Now, we can finally prove the convexity of the q-Catalan polynomials for $q \in (-1,0)$, which also establishes the validity of another hypothesis in Corollary 2.5.

Theorem 4.1. For $n \geq 2$, $C_n(q)$ is strictly convex in the range $-1 < q < 0$.

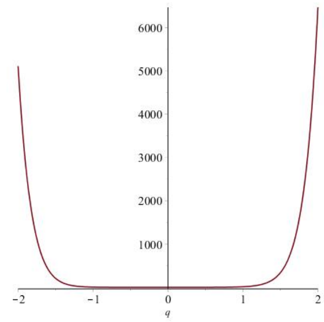

FIGURE 2. Convexity of $C_4(q)$.

Proof. Recall that $Q_n(q) = \sum_{j=2}^{n} \frac{jq^{j-1}}{1-q^j} - \sum_{j=n+2}^{2n} \frac{jq^{j-1}}{1-q^j}$, $C_n'(q) = C_n(q)Q_n(q)$ and $q^2Q_n'(q) = \sum_{j=2}^{n} \frac{jq^j(q^j-1+j)}{(1-q^j)^2} - \sum_{j=n+2}^{2n} \frac{jq^j(q^j-1+j)}{(1-q^j)^2}$. Our goal is to establish that $q^2Q_n'(q) > 0$ for $-1 < q < 0$, and hence $Q_n^2(q) + Q_n'(q) > 0$. The rest follows from the identity

$$C_n''(q) = (C_n(q)Q_n(q))' = C_n'(q)Q_n(q) + C_n(q)Q_n'(q) = C_n(q)(Q_n^2(q) + Q_n'(q))$$

(recall that $C_n(q) > 0$).

If n is an odd integer, then $q^2Q_n'(q) - q^2Q_{n-2}'(q) = L_n(q)$ and Lemma 3.6 reveals $L_n(q) > 0$. By induction on $n \geq 3$ and the fact that (for the base case) $q^2Q_3'(q) = L_3(q) > 0$, we firmly deduce that $q^2Q_n'(q) > 0$ for odd $n \geq 3$.

If n is even, from Lemma 3.5 we have $q^2Q_n'(q) - q^2Q_{n-1}'(q) = K_n(q) > 0$, and from the case n odd (just proven) we get $q^2Q_{n-1}'(q) > 0$ if $n \geq 4$. Hence $q^2Q_n'(q) > 0$ once again. The case $n = 2$ is easy, as $C_2(q) = 1 + q^2$. The proof is complete in all cases. □

Finally, we are ready to state (again) and prove the main result of this paper.

Theorem 4.2. The q-Catalan polynomials $C_n(q)$ are strictly convex over $\mathbb{R}$ for all $n \geq 2$.

Figure 2 portrays monotonicity and convexity of $C_4(q)$. Note also that $C_n(q) \geq C_n(0) = 1$.

Proof. The case $q > 0$ is handled by the fact that $C_n(q)$ has non-negative coefficients; see Proposition 2.1(d) and (e). The case $-1 < q < 0$ is the content of Theorem 4.1. The case $q < -1$ is implied by Corollary 2.5, Proposition 3.2 and Theorem 4.1. The special values for $q = 0$ and $q = -1$ appear in Proposition 2.1(a) and (c). The theorem follows. □

5. Concluding remarks

In this section, we leave the reader with certain inequalities of particular interest. Theorem 5.1 is in harmony with the preceding sections, it may also be regraded being of independent value. Corollary 5.3 displayed below seems *weaker* than Theorem 4.2, the main result of this paper; in the sense that it does not offer the extent as to *how large n* should be. So, it may be viewed principally as a theoretical contribution to the topic at hand.

Below, we say f_n defined on D *converges to f uniformly on compacta* if $f_n \to f$ uniformly on every compact subset $K \subset D$. Denote the interval $\{q \in \mathbb{R} : -1 < q < 1\}$ by $\mathcal{I}$.

Observe that, for $q \in \mathcal{I}$, MacMahon's q-Catalan polynomials $C_n(q)$ converge uniformly on compacta (for instance, by the Weierstrass M-test) to an infinite product, namely

$$\lim_{n\to\infty} C_n(q) = \lim_{n\to\infty} \prod_{k=2}^{n} \frac{1-q^{n+k}}{1-q^k} = \prod_{k=2}^{\infty} \frac{1}{1-q^k} := F(q),$$

which is a generating function for partitions $\lambda \vdash n$ with no part equal to 1.

On the other hand, the classical partition function $P(n)$ satisfies the so-called Hardy–Ramanujan–Rademacher (see [1], Chapter 5) asymptotic estimate $\log P(n) \sim \pi\sqrt{2n/3}$, and so

$$\lim_{n\to\infty} \frac{\log P(n)}{n} = 0.$$

In particular, the growth rate function of $P(n)$ is *subexponential*. Standard tests show that the infinite product and infinite series $G(q) = \prod_{k=1}^{\infty} \frac{1}{1-q^k} = \sum_{n=0}^{\infty} P(n)q^n$ have radius of convergence 1. Moreover, it is clear that

$$F(q) = \prod_{k=2}^{\infty} \frac{1}{1-q^k} = \sum_{n=0}^{\infty} (P(n) - P(n-1))q^n$$

shares the same interval of convergence $|q| < 1$. So, there is ample reason to study the function $F(q)$. To begin with, $F(q) > 0$ since each term in the product is such. Next, convergence allows computing derivatives freely over the interval $\mathcal{I}$, and we just do so by logarithmic differentiation:

$$F'(q) = F(q) \sum_{k=2}^{\infty} \frac{kq^{k-1}}{1-q^k}$$

and

$$F''(q) = F(q) \left(\sum_{k=2}^{\infty} \frac{kq^{k-1}}{1-q^k} \right)^2 + F(q) \sum_{k=2}^{\infty} \left(\frac{kq^{k-1}}{1-q^k} \right)^2 + F(q) \sum_{k=2}^{\infty} \frac{k(k-1)q^{k-2}}{1-q^k}.$$

After a geometric series expansion and infinite series manipulations, the last series takes the form

$$\begin{aligned}\sum_{k=2}^{\infty}\frac{k(k-1)q^{k-2}}{1-q^k} &= \sum_{k=2}^{\infty}k(k-1)q^{k-2}\sum_{n=0}^{\infty}q^{kn}\\ &= \sum_{n=0}^{\infty}q^{-2}\sum_{k=2}^{\infty}k(k-1)q^{k(n+1)}\\ &= \sum_{n=0}^{\infty}\frac{2q^{2n}}{(1-q^{n+1})^3}.\end{aligned}$$

Therefore, we arrive at

$$F''(q) = F(q)\left(\sum_{k=2}^{\infty}\frac{kq^{k-1}}{1-q^k}\right)^2 + F(q)\sum_{k=2}^{\infty}\left(\frac{kq^{k-1}}{1-q^k}\right)^2 + F(q)\sum_{k=1}^{\infty}\frac{2q^{2k-2}}{(1-q^k)^3}.$$

From here, we may readily infer

Theorem 5.1. $F(q)$ is strictly convex in the interval $\mathcal{I}$, i.e. $F''(q) > 0$.

Note 5.2. The same statement (with the same proof) also holds for the generating function $G(q) = \prod_{k=1}^{\infty}\frac{1}{1-q^k}$.

Corollary 5.3. For each compact subset $S \subset \mathcal{I}$, if n is large enough, then $C_n(q)$ is strictly convex in S.

Proof. Since the polynomials $C_n(q)$ converge uniformly on compacta to the analytic function $F(q)$, it follows that $C_n''(q) \to F''(q)$ uniformly on compacta. Since $F''(q)$ is a continuous function on the compact set S, it has a minimum value $m^* > 0$ there. By uniform convergence, there must be an n_0 such that $\sup_{q\in S}|C_n''(q)-F''(q)| \le \frac{m^*}{2}$ for all $n \ge n_0$. By triangular inequality, we must therefore have $C_n''(q) \ge \frac{m^*}{2} > 0$ for all $q \in S$ and $n \ge n_0$. □

Note 5.4. Although we have arrived at the first proof for the convexity of the q-Catalan polynomials, our method stands as highly technical. As history has shown us abundantly, alternative and more concise methods usually come to replace initial attempts. We hope to see such follow-ups concerning our main result.

For example, *is there a more elegant proof of Lemma 3.6 than what this paper offers?*

On a more general note, it would be interesting to develop more widely applicable techniques to prove convexity and similar properties for other families of q-polynomials arising in Combinatorics or elsewhere.

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Conflict of Interest. On behalf of all authors, the corresponding author states that there is no conflict of interest.

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