GENERALIZED CUBIC PARTITIONS

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Abstract. A cubic partition consists of partition pairs (λ, μ) such that $|\lambda| + |\mu| = n$ where μ involves only even integers but no restriction is placed on λ. This paper initiates the notion of generalized cubic partitions and will prove a number of new congruences akin to the classical Ramanujan-type. The tools emphasize three methods of proofs. The paper concludes with a conjecture on the rarity of the aforementioned Ramanujan-type congruences.

1. Introduction

Consider the set $\mathcal{G}_c(n)$ of all partitions of an integer $n \geq 1$ such that each even part comes in c different colors. We call these *generalized cubic partitions*. Denote the enumeration of $\mathcal{G}_c(n)$ by $a_c(n)$ under the convention $a_c(0) := 1$. One can associate an Euler-type generating function

$$\sum_{n\geq 0} a_c(n)\, q^n = \prod_{k\geq 1} \frac{1}{(1-q^k)(1-q^{2k})^{c-1}}. \tag{1.1}$$

Notice that $a_1(n) = p(n)$ the usual (unrestricted) partition of n, while $a_2(n)$ enumerates the so-called *cubic partitions* of n. As examples for the latter, $a_2(2) = \#\{2, 2, 11\} = 3, a_2(4) = \#\{4, 4, 31, 22, 22, 22, 211, 211, 1111\} = 9$.

Let us adopt the notation $(a;q)_\infty = \prod_{k\geq 0}(1-aq^k)$. Among Ramanujan's discoveries, the following identity (for the classical partition function)

$$\sum_{n\geq 0} p(5n+4)\, q^n = 5\,\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6},$$

is regarded as his "Most Beautiful Identity" by both Hardy and MacMahon (see [9, p. xxxv]). Based on an identity on Ramanujan's cubic continued fractions [1], Chan [4] introduced the notion of cubic partitions, denoted $a_2(n)$ above, and succeeded in establishing the following elegant analogue

$$\sum_{n\geq 0} a_2(3n+2)\, q^n = 3\,\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^2;q^2)_\infty^4}.$$

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It is immediate [4] that $a_2(3n+2) \equiv 0 \pmod 3$. Since then many authors studied similar congruences for $c_2(n)$ (see [5], [6] and references therein).

Recall the celebrated Ramanujan congruences [9, p. 210, p. 230] for the partition function

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod 5, \\ p(7n+5) &\equiv 0 \pmod 7, \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{1.2}$$

By analogy, here are our main results.

Theorem 1.1. For all integers $n \geq 0$, the following congruences hold true.

(I) $a_2(3n+2) \equiv 0 \pmod 3$,
(II) $a_4(5n+4) \equiv 0 \pmod 5$,
(II)′ $a_4(5n+2) \equiv 0 \pmod 5$,
(III) $a_3(7n+4) \equiv 0 \pmod 7$,
(IV) $a_5(11n+10) \equiv 0 \pmod{11}$.

This paper is organized as follows. Section 2 highlights the basic results from the theory of modular forms to pave the way for Sections 4 and 5. In Section 3 we present the first set of elementary proofs for most of our results in Theorem 1.1. Section 4 contains another proof of Theorem 1.1 (I) which is less elementary than the approach taken in Section 3. Among the assertions in Theorem 1.1, the one pertaining to a congruence modulo 11 turns out to require the theory of modular forms be brought to bear. This is precisely the content of Section 5. Finally, in Section 6, we summarize this paper with one more application of our methods and an open problem.

2. BACKGROUND: MODULAR FORMS

In this section, we need some definitions and basic facts on modular forms that are instrumental in furnishing our proof of Theorem 1.1 (IV). For additional details, see for example [7, 8]. We first identify the matrix groups

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\},$$

$$\Gamma_\infty := \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\},$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod N \right\},$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\},$$

where N is a positive integer. A subgroup Γ of the group $\mathrm{SL}_2(\mathbb{Z})$ is called a *congruence subgroup* if $\Gamma(N) \subseteq \Gamma$ for some N. The smallest N such that $\Gamma(N) \subseteq \Gamma$ is called the *level of* Γ. For example, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level N.

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half of the complex plane. Then, the following subgroup of the general linear group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \dfrac{az+b}{cz+d}$. We identify ∞ with $\dfrac{1}{0}$ and define

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{r}{s} = \frac{ar+bs}{cr+ds},$$

where $\dfrac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A *cusp* of Γ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of Γ. The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and ℓ is an integer, then define the slash operator $|_\ell$ by

$$(f|_\ell \gamma)(z) := (\det \gamma)^{\ell/2} (cz+d)^{-\ell} f(\gamma z).$$

Definition 2.1. Let Γ be a congruence subgroup of level N. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *modular* form with integer weight ℓ on Γ if the following hold:

(1) We have
$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^\ell f(z)$$
for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$.

(2) If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_\ell \gamma)(z)$ has a Fourier expansion of the form
$$(f|_\ell \gamma)(z) = \sum_{n \geq 0} a_\gamma(n) q_N^n,$$
where $q_N := e^{2\pi i z/N}$.

For a positive integer ℓ, the complex vector space of modular forms of weight ℓ with respect to a congruence subgroup Γ is denoted by $M_\ell(\Gamma)$.

Definition 2.2. [8, Definition 1.15] If χ is a Dirichlet character modulo N, then we say that a modular form $f \in M_\ell(\Gamma_1(N))$ has *Nebentypus character* χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_\ell(\Gamma_0(N), \chi)$.

In this paper, the relevant modular forms are those that arise from eta-quotients. The *Dedekind eta-function* $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi iz}$ and $z \in \mathbb{H}$, the upper half-plane. A function $f(z)$ is called an *eta-quotient* if it is of the form

$$f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta},$$

where N is a positive integer and r_δ is an integer. We now recall two valuable theorems from [8, p. 18] which will be used to prove our results.

Theorem 2.3. [8, Theorem 1.64] *If $f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}$ is an eta-quotient such that $\ell = \frac{1}{2}\sum_{\delta \mid N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^\ell f(z)$$

for every $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. Here the character χ is defined by $\chi(\bullet) := \left(\frac{(-1)^\ell s}{\bullet}\right)$, where $s := \prod_{\delta \mid N} \delta^{r_\delta}$.

Suppose that f is an eta-quotient satisfying the conditions of Theorem 2.3 and that the associated weight ℓ is a positive integer. If the function $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N), \chi)$. The next theorem gives a necessary condition for determining orders of an eta-quotient at cusps.

Theorem 2.4. [8, Theorem 1.65] *Let c, d and N be positive integers with $d \mid N$ and $\gcd(c, d) = 1$. If f is an eta-quotient satisfying the conditions of Theorem 2.3 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta \mid N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta}.$$

We now remind ourselves a result of Sturm [10] which gives a criterion to test whether two modular forms are congruent modulo a given prime.

Theorem 2.5. *Let p be a prime number, and $f(z) = \sum_{n=n_0}^{\infty} a(n)q^n$ and $g(z) = \sum_{n=n_1}^{\infty} b(n)q^n$ be modular forms of weight k for $\Gamma_0(N)$ of characters χ and ψ, respectively, where $n_0, n_1 \geq 0$. If either $\chi = \psi$ and*

$$a(n) \equiv b(n) \pmod{p} \ \textit{for all} \ n \leq \frac{kN}{12} \prod_{d \ \textit{prime};\ d \mid N} \left(1 + \frac{1}{d}\right),$$

or $\chi \neq \psi$ and

$$a(n) \equiv b(n) \pmod{p} \ \textit{for all} \ n \leq \frac{kN^2}{12} \prod_{d \ \textit{prime};\ d \mid N} \left(1 - \frac{1}{d^2}\right),$$

then $f(z) \equiv g(z) \pmod{p}$ (i.e., $a(n) \equiv b(n) \pmod{p}$ for all n).

We now recall the definition of Hecke operators. Let m be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$. Then the action of Hecke operator T_m on $f(z)$ is defined by

$$f(z) \mid T_m := \sum_{n=0}^{\infty} \left(\sum_{d \mid \gcd(n,m)} \chi(d) d^{\ell-1} a\left(\frac{nm}{d^2}\right)\right) q^n.$$

In particular, if $m = p$ is prime, we have

$$f(z) \mid T_p := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p) p^{\ell-1} a\left(\frac{n}{p}\right)\right) q^n. \tag{2.1}$$

We take by convention that $a(n/p) = 0$ whenever $p \nmid n$. The next result follows directly from (2.1).

Proposition 2.6. Let p be a prime, $g(z) \in \mathbb{Z}[[q]], h(z) \in \mathbb{Z}[[q^p]]$, and $k > 1$. Then

$$(g(z)h(z)) \mid T_p \equiv \left(g(z) | T_p \cdot h(z/p)\right) \pmod{p}.$$

3. Proof of Theorem 1.1

In the current section, we exhibit a simpler proof for each of congruences (I), (II), (II)′ and (III) from Theorem 1.1.

Let's adopt some notations: $f(q) := \prod_{k\geq 1}(1-q^k)$ and $g(q) := \prod_{k\geq 1}(1+q^k)$. Then, the generating function in (1.1) can be presented as

$$F_c(q) := \frac{1}{f(q)\,f(q^2)^{c-1}}.$$

Proof of (I). Rewrite $F_2(q)$ and compute modulo 3 to get that

$$\begin{aligned} F_2(q) &= \frac{f^2(q)\,f^2(q^2)}{f^3(q)\,f^3(q^2)} \equiv \frac{f^2(q)\,f^2(q^2)}{f(q^3)\,f(q^6)} = \frac{f^3(q)\,g(q)\,f(q^2)}{f(q^3)\,f(q^6)} \\ &\equiv \frac{f(q^3)\,g(q)\,f(q^2)}{f(q^3)\,f(q^6)} = \frac{g(q)\,f(q^2)}{f(q^6)} \pmod 3. \end{aligned}$$

On the other hand, *Jacobi's triple product* (see [2, p. 35, Entry 19]) may be stated in the manner

$$(q;q)_\infty\,(-z^{-1};q)_\infty\,(-zq;q)_\infty = \sum_{n\in\mathbb{Z}} q^{\binom{n+1}{2}}\,z^n. \tag{3.1}$$

Choosing $z=1$ leads to the identity $g(q)\,f(q^2) = \sum_{n\geq 0} q^{\binom{n+1}{2}}$ and hence

$$F_2(q) \equiv \frac{\sum_{n\geq 0} q^{\binom{n+1}{2}}}{f(q^6)} \pmod 3. \tag{3.2}$$

Since $\binom{n+1}{2} \equiv 0$ or 1 modulo 3, we infer that the coefficients of q^{3n+2} in the expansion (3.2) all *vanish*. That means, $a_2(3n+2) \equiv 0 \pmod 3$.

Proof of (II) and (II)′. Computing modulo 5 and invoking equation (3.1), we obtain

$$\begin{aligned} F_4(q) &= \frac{f^4(q)\,f^2(q^2)}{f^5(q)\,f^5(q^2)} \equiv \frac{f^4(q)\,f^2(q^2)}{f(q^5)\,f(q^{10})} = \frac{f^5(q)\,g(q)\,f(q^2)}{f(q^5)\,f(q^{10})} \\ &\equiv \frac{f(q^5)\,g(q)\,f(q^2)}{f(q^5)\,f(q^{10})} = \frac{g(q)\,f(q^2)}{f(q^{10})} = \frac{\sum_{n\geq 0} q^{\binom{n+1}{2}}}{f(q^{10})} \pmod 5. \end{aligned}$$

Because $\binom{n+1}{2} \equiv 0, 1$ or 3 modulo 5, we gather that the coefficients of q^{5n+4} and q^{5n+2}, in $F_4(q)$, are all zeros. This proves (II) and (II)′.

Proof of (III). Mimicking the above two proofs, we proceed to find

$$\begin{aligned} F_3(q) &= \frac{f^6(q)\,f^5(q^2)}{f^7(q)\,f^7(q^2)} \equiv \frac{f^6(q)\,f^5(q^2)}{f(q^7)\,f(q^{14})} \equiv \frac{g(q)\,f^4(q^2)}{f(q^{14})} \\ &= \frac{f^3(q^2)\,g(q)\,f(q^2)}{f(q^{14})} \equiv \frac{f^3(q^2)\cdot\sum_{n\geq 0} q^{\binom{n+1}{2}}}{f(q^{14})} \pmod 7. \end{aligned}$$

Next, we employ Jacobi's identity (3.1) to produce the relation [3, p. 14]

$$f^3(q^2) = \sum_{n\geq 0}(-1)^n(2n+1)q^{2\binom{n+1}{2}},$$

and as a result there holds true that

$$F_3(q) \equiv \frac{\sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)} \cdot \sum_{n\geq 0} q^{\binom{n+1}{2}}}{f(q^{14})} \pmod{7}.$$

We focus on two congruences for the exponents in the numerator:

$$\frac{n(n+1)}{2} \equiv 0,1,3,6 \pmod{7} \qquad \text{and} \qquad n(n+1) \equiv 0,2,5,6 \pmod{7}.$$

If we split up the congruence class then the sums can be written as

$$\sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)} = T_0 + T_2 + T_5 + T_6 \qquad \text{and}$$

$$\sum_{n\geq 0} q^{\binom{n+1}{2}} = S_0 + S_1 + S_3 + S_6.$$

However, the coefficients in T_5 arrive from $2n+1$ which are precisely zero modulo 7 when $n(n+1) \equiv 5 \pmod{7}$. Therefore, we end up with

$$F_3(q) \equiv \frac{(T_0 + T_2 + T_6)(S_0 + S_1 + S_3 + S_6)}{f(q^{14})} \pmod{7}.$$

Direct calculation shows that no monomial of the type q^{7n+4} can appear in the numerator, and this proves the congruence $a_3(7n+4) \equiv 0 \pmod{7}$. □

4. A second proof for Theorem 1.1(i)

In this section, we offer yet another elementary proof of the first congruence in Theorem 1.1. To this end, let's start with a basic observation. If $\xi = e^{\frac{2\pi i}{3}}$ is a primitive 3^{rd} root of unity then

$$\prod_{s=0}^{2}(1 - q^n\xi^{ns}) = \begin{cases} 1 - q^{3n}, & \text{if } \gcd(n,3) = 1 \\ (1-q^n)^3, & \text{if } 3 \mid n \end{cases} \tag{4.1}$$

which in turn implies that

$$\prod_{n\geq 1}\prod_{s=0}^{2}(1 - q^n\xi^{ns}) = \frac{f^4(q^3)}{f(q^9)} \qquad \text{or} \qquad \frac{1}{f(q)} = \frac{f(q^9)}{f^4(q^3)}\prod_{n\geq 1}\prod_{s=1}^{2}(1 - q^n\xi^{ns}).$$

Lemma 4.1. *It is true that*

$$f(q)\,f(q^2) = \frac{(-q^3;q^3)_\infty f^2(q^9)}{(-q^9;q^9)_\infty^2} - q\,f(q^9)\,f(q^{18}) - 2q^2\,\frac{f^2(q^{18})\,(-q^9;q^9)_\infty}{(-q^3;q^3)_\infty}.$$

Proof. The Dedekind's eta-function $\eta(z)$ enables us to convert the task into an equivalent formulation:

$$\begin{aligned}\eta(8z)\,\eta(16z) &= \frac{\eta(48z)\,\eta^4(72z)}{\eta(24z)\,\eta^2(144z)} - \eta(72z)\,\eta(144z) - 2\,\frac{\eta(24z)\,\eta^4(144z)}{\eta(48z)\,\eta^2(72z)} \\ &:= G_0(z) + G_1(z) + G_2(z).\end{aligned}$$

We now use Theorems 2.3 and 2.4 to ensure that $\eta(8z)\,\eta(16z)$ and each $G_j(z) \in M_1(\Gamma_0(1152), (\frac{-2}{\bullet}))$, for $j \in \{0, 1, 2\}$, are modular forms. Thus, Sturm's Theorem 2.5 can be brought to bear which shows that it suffices to check equality of the declared identity holds for 192 coefficients on both sides of the above eta equations. The latter claim we confirm with the help of a symbolic software and thereby completing the proof. $\square$

Lemma 4.2. *Let $E(q) := f^3(q) + 3q\,f^3(q^9)$. Then, we have that*

$$F_2(q) = \frac{f^3(q^9)E(q^2) + q\,f^3(q^{18})\,E(q) + 3q^2 f^3(q^9)\,f^3(q^{18})}{f^4(q^3)\,f^4(q^6)}$$

Proof. Lemma 4.1 is a *trisection* of $U(q) := f(q)\,f(q^2)$ according to the congruences modulo 3 to write as $U(q) = U_0(q)+U_1(q)+U_2(q)$, where $U_j(q)$ consists of monomials q^{3m+j}. More specifically, $U_1(q) = -qf(q^9)\,f(q^{18})$,

$$U_0(q) = \frac{(-q^3;q^3)_\infty f^2(q^9)}{(-q^9;q^9)^2_\infty} \qquad \text{and} \qquad U_2(q) = -2q^2\,\frac{f^2(q^{18})\,(-q^9;q^9)_\infty}{(-q^3;q^3)_\infty}.$$

Similarly, let $F_2(q) = \frac{1}{U(q)} = P_0(q) + P_1(q) + P_2(q)$. Upon multiplying these two expansions and combining based on powers of q modulo 3, we arrive at the system

$$\begin{aligned} U_0P_0 + U_2P_1 + U_1P_2 &= 1 \\ U_1P_0 + U_0P_1 + U_2P_2 &= 0 \\ U_2P_0 + U_1P_1 + U_0P_2 &= 0. \end{aligned}$$

By standard tools, we obtain that $P_j(q) = \frac{D_j(q)}{D(q)}$ for $j \in \{0, 1, 2\}$, where $D_0 = U_0^2 - U_1U_2, D_1 = U_2^2 - U_0U_1$,

$$D_2 = \det\begin{pmatrix} U_1 & U_0 \\ U_2 & U_1 \end{pmatrix} = U_1^2 - U_0U_2 \qquad \text{and} \qquad D(q) = \det\begin{pmatrix} U_0 & U_2 & U_1 \\ U_1 & U_0 & U_2 \\ U_2 & U_1 & U_0 \end{pmatrix}.$$

Let $\xi = e^{\frac{2\pi i}{3}}$. The determinant $D(q)$ involves a circulant matrix and hence its value is the product of eigenvalues

$$D(q) = \prod_{s=0}^{2}(U_0 + \xi^s U_1 + \xi^{2s} U_2) = \prod_{s=0}^{2}(U_0(\xi^s q) + U_1(\xi^s q) + U_2(\xi^s q))$$
$$= \prod_{s=0}^{2} U(\xi^s q) = \prod_{s=0}^{2} f(\xi^s q)\, f(\xi^{2s} q^2) = \frac{f^4(q^3)\, f^4(q^6)}{f(q^9)\, f(q^{18})}.$$

In the last step, we made use of the product formula from (4.1).

It is rather clear from the above-mentioned explicit formulas that we can obtain $U_1(q) = -qU(q^9)$ and $U_0(q)U_2(q) + 2U_1^2(q) = 0$. These imply $D_2(q) = U_1^2 - U_0 U_2 = 3U_1^2 = 3q^2 f^2(q^9)\, f^2(q^{18})$ and hence, we obtain

$$P_2(q) = \frac{D_2(q)}{D(q)} = 3\, q^2 \cdot \frac{f^3(q^9)\, f^3(q^{18})}{f^4(q^3)\, f^4(q^6)}. \tag{4.2}$$

A similar argument and analysis produces

$$P_0(q) = \frac{f^3(q^9)}{f^4(q^3)\, f^4(q^6)} \cdot E(q^2) \qquad \text{and} \qquad P_1(q) = q\, \frac{f^3(q^{18})}{f^4(q^3)\, f^4(q^6)} \cdot E(q).$$

The proof is now complete. □

Proof of Theorem 1.1 (I). By (4.2) in the proof of Lemma 4.2, we infer that

$$P_2(q) = \sum_{n\geq 0} a_2(3n+2) q^{3n+2} = 3\, q^2 \cdot \frac{f^3(q^9)\, f^3(q^{18})}{f^4(q^3)\, f^4(q^6)}$$

which is ample evidence to help us conclude $a_2(3n + 2) \equiv 0 \pmod 3$. □

5. Proof for the congruence modulo 11

We are finally ready to supply the proof for Theorem 1.1 (IV).

Proof of Theorem 1.1 (IV). By (1.1), we have

$$\sum_{n\geq 0} a_5(n)\, q^n = \prod_{k\geq 1} \frac{1}{(1-q^k)(1-q^{2k})^4}. \tag{5.1}$$

Let

$$G(z) := \frac{\eta^{32}(z)}{\eta^4(2z)}.$$

By Theorems 2.3 and 2.4, we find that $G(z)$ is a modular forms of weight 14, level 4 and character $\chi_0 = (\frac{2^{-4}}{\bullet})$. By (5.1), the Fourier expansions of our form satisfy

$$G(z) = \left(\sum_{n=0}^{\infty} a_5(n) q^{n+1}\right) \prod_{k\geq 1}(1-q^k)^{33}.$$

Using Proposition 2.6, we calculate that

$$G(z)\,|\,T_{11} \equiv \left(\sum_{n=0}^{\infty} a_5(11n+10)q^{n+1}\right)\prod_{k\geq 1}(1-q^k)^3 \pmod{11}.$$

Since the Hecke operator is an endomorphism on $M_{14}(\Gamma_0(4),\chi_0)$, we have that $G(z)\,|\,T_{11} \in M_{14}(\Gamma_0(4),\chi_0)$. By Theorem 2.5, the Sturm bound for this space of forms is 7. Hence, Theorem 2.5 confirms that $G(z)\,|\,T_{11} \equiv 0 \pmod{11}$. This completes the proof of the congruence. □

6. Conclusion

In this section, we bring in a slight variation of the cubic partitions and apply the techniques from Section 4 as a further illustration. Finally, we leave the reader with an open problem.

Consider the generating function for *pairs of cubic partitions*

$$h(q) := \frac{1}{f^2(q)\,f^2(q^2)}.$$

Lemma 6.1. *We have that*

$$f^2(q)\,f^2(q^2) = \frac{f^3(q^2)\,(-q^4;q^4)_\infty^2\,f(q^8)}{(-q^8;q^8)_\infty^2} - 2q\,f^3(q^2)\,(-q^8;q^8)_\infty\,f(q^{16}).$$

Proof. The identity in question translates to the eta-quotient

$$\eta^2(4z)\,\eta^2(8z) = \frac{\eta^3(8z)\,\eta^5(32z)}{\eta^2(64)\,\eta^2(16z)} - 2\,\frac{\eta^3(8z)\,\eta^2(64z)}{\eta(32z)}.$$

Each of the three eta-quotients involved here is a modular form belonging to $M_2(\Gamma_0(64),(\frac{1}{\bullet}))$. Hence, Sturm's Theorem 2.5 applies with the bound 16 (that is as many coefficients we need to check for agreement). The proof ends. □

Theorem 6.2. *If $f^2(q)\,f^2(q^2) = V_0(q)+V_1(q)$ and $h(q) = P_0(q)+P_1(q)$ then*

$$P_0(q) = V_0(q)\cdot h(q)\,h(-q) \qquad \textit{and} \qquad P_1(q) = -V_1(q)\cdot h(q)\,h(-q).$$

Proof. From the bisection given in Lemma 6.1, we reckon that

$$V_0 = \frac{f^3(q^2)\,(-q^4;q^4)_\infty^2\,f(q^8)}{(-q^8;q^8)_\infty^2} \quad \text{and} \quad V_1 = -2q\,f^3(q^2)\,(-q^8;q^8)_\infty\,f(q^{16}).$$

Next, solve the system of equations

$$\begin{aligned} V_0P_0 + V_1P_1 &= 1\\ V_1P_0 + V_0P_1 &= 0 \end{aligned}$$

for P_0 and P_1. Observing that $\det\begin{pmatrix} V_0 & V_1\\ V_1 & V_0\end{pmatrix} = \frac{1}{h(q)\,h(-q)}$, the proof follows. □

Iterating the relation $h(q) = (V_0(q) - V_1(q))h(q)h(-q)$ of Theorem 6.2, we arrive at the next result.

Corollary 6.1. For integers $\ell \geq 0$, we have

$$h(q) = [V_0^2(q) - V_1^2(q)]^{5\cdot 2^\ell - 1} \cdot (V_0(q) - V_1(q)) \cdot [h(q)]^{5\cdot 2^\ell} \cdot [h(-q)]^{5\cdot 2^\ell}.$$

To produce our next application, we need to recall Ramanujan's two-variable theta-function defined as

$$f(a,b) := \sum_{n\in\mathbb{Z}} a^{\frac{n(n+1)}{2}} b^{\binom{n}{2}}.$$

One special case would then be $f(-q,-q^2) = f(q) = \prod_{k\geq 1}(1-q^k)$, for which there is a 5-dissection $f(q) = A_0(q) - qA_1(q) - q^2A_2(q)$ where

$$A_0(q) := \frac{f(q^{25})\, f(-q^{15},-q^{10})}{f(-q^{20},-q^5)}, \quad A_1(q) := f(q^{25}), \quad A_2(q) := \frac{A_1^2(q)}{A_0(q)},$$

due to Ramanujan (for instance, see Berndt's book [2, pp. 81-82]). Thus,

$$f^3(q) = (A_0 - 3A_1A_2q^5) - q\,(3A_0^2A_1 + A_2^3q^5) + 5q^3A_1^3$$

from which it is immediate that the 5-dissection of the product $f^3(q)f^3(q^2)$ assumes the form $B_0(q) + qB_1(q) + q^2B_2(q) + q^3B_3(q) + 25q^4B_4(q)$. As a consequence, the coefficients of q^{5n+4}, in $f^3(q)f^3(q^2)$, vanish modulo 25. In particular, if $h(q) = \sum_{n\geq 0} b(n)\, q^n$ then we recover the following congruence [11, (3.7) in Thm. 3.2]

$$b(5n+4) \equiv 0 \pmod 5 \qquad \text{for all } n \geq 0$$

because we may rewrite (applying the binomial theorem)

$$h(q) = \frac{f^3(q)f^3(q^2)}{f^5(q)f^5(q^2)} \equiv \frac{f^3(q)f^3(q^2)}{f(q^5)f(q^{10})} \pmod 5.$$

Finally, in a true tradition of the famous Ramanujan's congruences, we now declare the following claim.

Conjecture 6.2. There are no Ramanujan-type congruences (1.2) for $a_c(n)$ apart from those reducible to Ramanujan's (1.2) or our Theorem 1.1.

Remark 6.3. The exemptions (values of c) in the above conjecture should be understood as congruences, modulo a prime p, that appear in the form

$$F_c(q) \equiv F_1(q) \prod_{k\geq 1} \frac{1}{(1-q^{2p})^r} \qquad \text{when } p \in \{5,7,11\}, \text{ or}$$

$$F_c(q) \equiv F_b(q) \prod_{k\geq 1} \frac{1}{(1-q^{2p})^s} \quad \text{when } (b,p) \in \{(2,3),(4,5),(3,7),(5,11)\},$$

for some integers r and s.

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