A CONGRUENCE FOR A DOUBLE HARMONIC SUM

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Abstract. In this short note, our primary purpose is to prove the congruence

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{(-1)^k}{k} \sum_{j=\lfloor \frac{k}{2} \rfloor+1}^{k} \frac{1}{2j-1} \equiv 0 \mod p.$$

Along the way, a number of auxiliary results of independent interest are found.

1. Introduction

The main target and motivation for this work is the present authors' intent to respond to a certain challenge proposed at the public forum called `Mathoverflow` in which the proposer asks a proof for the congruence

$$\sum_{k=1}^{\frac{p-1}{2}} \frac{(-1)^k}{k} \sum_{j=\lfloor \frac{k}{2} \rfloor+1}^{k} \frac{1}{2j-1} \equiv 0 \mod p.$$

After some effort, we succeed in doing so.

The following notations and conventions will be adhered to throughout the discussion.

Let $p \geq 5$ be a prime. Denote $p' = \frac{p-1}{2}, p'' = \lfloor \frac{p-1}{4} \rfloor$ and the Fermat's quotients by $q_2 = \frac{2^{p-1}-1}{p}$ while $\left(\frac{a}{p}\right)$ stands for the Legendre's symbol. For brevity, $\equiv_p$ designates congruence modulo p. The Euler numbers E_n are defined by the exponential generating function

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}.$$

The generalized harmonic numbers are given by $H_n(a) = \sum_{j=1}^{n} \frac{1}{n^a}$ so that the classical harmonic numbers become $H_n = H_n(1)$.

The organization of the paper is as follows. In Section 2, we list some relevant congruences for harmonic numbers which appear in the existing literature. In Section 3, we prove a few preparatory statements to conclude with our main result as advertised at the beginning of this section and the Abstract. In Section 4, we show an evaluation of a related definite sum

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{j=\lfloor \frac{k}{2} \rfloor+1}^{k} \frac{1}{2j-1} = -\frac{5\pi^2}{48}.$$

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2. Background results

In this section, we shall recall certain well-known congruences which play a direct roll in the sequel. The most basic result in this direction is that of Wolstenholme's $H_{p-1} \equiv_p 0$, which has been strengthened since.

Lemma 1. *(Wolstenholme)*

$$H_{p-1} \equiv_{p^2} 0. \tag{1}$$

Lemma 2. *(Eisenstein)*

$$H_{p'} \equiv_p -2q_2. \tag{2}$$

Lemma 3. *We have the elementary congruences*

$$\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv_p 0, \qquad \sum_{k=1}^{p'} \frac{1}{k^2} \equiv_p 0 \qquad \textit{and} \qquad \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \equiv_p 0. \tag{3}$$

Proof. The first congruence is clear from $\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv_p \sum_{k=1}^{p-1} k^2 = \frac{p(p-1)(2p-1)}{6} \equiv_p 0$. With the change of variables $k \to p' - k + 1$, we obtain

$$\sum_{k=1}^{p-1} \frac{1}{(2k-1)^2} = \sum_{k=1}^{p'} \frac{1}{(2(p'-k+1)-1)^2} \equiv_p \frac{1}{4} \sum_{k=1}^{p'} \frac{1}{k^2}$$

which implies that

$$0 \equiv_p \sum_{k=1}^{p-1} \frac{1}{k^2} = \frac{1}{4} \sum_{k=1}^{p'} \frac{1}{k^2} + \sum_{k=1}^{p-1} \frac{1}{(2k-1)^2} \equiv_p \frac{1}{2} \sum_{k=1}^{p'} \frac{1}{k^2}.$$

and also

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} = \frac{1}{4} \sum_{k=1}^{p'} \frac{1}{k^2} - \sum_{k=1}^{p'} \frac{1}{(2k-1)^2} \equiv_p \frac{1}{4} \sum_{k=1}^{p'} \frac{1}{k^2} - \frac{1}{4} \sum_{k=1}^{p'} \frac{1}{k^2} = 0.$$

The proof is complete. □

Lemma 4. *(Glaisher)* [2, (43)]

$$H_{p''} \equiv_p -3q_2. \tag{4}$$

Lemma 5. [2, (19) and (20)]

$$H_{p'}(2) \equiv_p 0 \quad \textit{and} \quad H_{p''}(2) \equiv_p 4(-1)^{p'} E_{p-3}. \tag{5}$$

Hence,

Lemma 6.

$$\sum_{k=1}^{p-1} \frac{H_k}{k} = \frac{1}{2} \left(H_{p-1}^2 - H_{p-1}(2) \right) \equiv_p 0, \tag{6}$$

$$\sum_{k=1}^{p'} \frac{H_k}{k} = \frac{1}{2} \left(H_{p'}^2 - H_{p'}(2) \right) \equiv_p 2q_2^2, \tag{7}$$

$$\sum_{k=1}^{p''} \frac{H_k}{k} = \frac{1}{2} \left(H_{p''}^2 + H_{p''}(2) \right) \equiv_p \frac{9q_2^2}{2} + 2(-1)^{p'} E_{p-3}. \tag{8}$$

Lemma 7. *Define the function* $\pounds_d(x) = \sum_{j=1}^{p-1} \frac{x^j}{j^d}$. *Then,*

$$\sum_{j=1}^{p-1} \frac{x^j H_j}{j} \equiv_p \pounds_2(x) - \pounds_2(1-x). \tag{9}$$

Proof. We have

$$\sum_{j=1}^{p-1} \frac{(1-x)^j - 1}{j} = \sum_{j=1}^{n} \binom{p-1}{j} \frac{(-x)^j}{j} \equiv_{p^2} \sum_{j=1}^{p-1} \frac{x^j}{j}(1 - pH_j)$$

which implies that

$$\sum_{j=1}^{p-1} \frac{x^j H_j}{j} \equiv_p \frac{\pounds_1(x) - \pounds_1(1-x) + H_{p-1}}{p} \equiv_p \pounds_2(x) - \pounds_2(1-x).$$

where we used [1, (6)] and (1), as well as

$$-\pounds_2(x) \equiv_p \frac{1}{p}\left(\frac{x^p + (1-x)^p - 1}{p} + \pounds_1(1-x)\right).$$

□

Lemma 8. *We have*

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} H_k \equiv_p q_2^2. \tag{10}$$

Proof. By Lemma 7, the last congruence in (3) and [1, (2)], we gather that

$$\sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k} \equiv_p \pounds_2(-1) - \pounds_2(2) \equiv_p q_2^2. \tag{11}$$

□

Lemma 9. *We have*

$$\sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} \equiv_p \frac{q_2^2}{2} + (-1)^{p'} E_{p-3}. \tag{12}$$

Proof. We proceed as follows:

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k} &= \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} + \sum_{k=p'+1}^{p-1} \frac{(-1)^k H_k}{k} = \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} + \sum_{k=1}^{p'} \frac{(-1)^{p-k} H_{p-k}}{p-k} \\ &\equiv_p \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} + \sum_{k=1}^{p'} \frac{(-1)^k H_{p-k}}{k} \equiv_p \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} + \sum_{k=1}^{p'} \frac{(-1)^k H_{k-1}}{k} \\ &= 2\sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} - \sum_{k=1}^{p'} \frac{(-1)^k}{k^2}\end{aligned}$$

where we used the fact that $H_{p-k} \equiv_p H_{k-1}$. Hence, by (5) and (11),

$$\begin{aligned}\sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} &\equiv_p \frac{1}{2}\sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k} + \frac{1}{2}\sum_{k=1}^{p'} \frac{(-1)^k}{k^2} \\ &= \frac{1}{2}\sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k} + \frac{1}{2}\left[-H_{p'}(2) + \frac{1}{2}H_{p''}(2)\right] \\ &\equiv_p \frac{q_2^2}{2} + (-1)^{p'} E_{p-3}.\end{aligned}$$

□

3. Main Results

In order to reach the main goal of this paper, we first establish a series of crucial preparatory statements. From (6) and (11), it is immediate that

$$\sum_{k=1}^{p'} \frac{H_{2k}}{k} = \sum_{k=1}^{p-1} \frac{H_k}{k} + \sum_{k=1}^{p-1} \frac{(-1)^k H_k}{k} \equiv_p q_2^2. \tag{13}$$

On the other hand, (7) and (12) lead to

$$\sum_{k=1}^{p''} \frac{H_{2k}}{k} = \sum_{k=1}^{p'} \frac{H_k}{k} + \sum_{k=1}^{p'} \frac{(-1)^k H_k}{k} \equiv_p \frac{5q_2^2}{2} + (-1)^{p'} E_{p-3}. \tag{14}$$

Lemma 10. *We have*

$$\sum_{k=1}^{p'} \frac{(-1)^k H_{2k}}{k} \equiv_p \frac{q_2^2}{4}. \tag{15}$$

Proof. By [3, Section 4]

$$\begin{aligned}\sum_{k=1}^{p'} \frac{(-1)^k H_{2k}}{k} &= 2\sum_{k=1}^{p'} \frac{(i^2)^k H_{2k}}{2k} = 2\,\mathrm{Re}\left(\sum_{k=1}^{p-1} \frac{i^k H_k}{k}\right) \\ &\equiv_p 2\,\mathrm{Re}\left(\pounds_2(i) - \pounds_2(1-i)\right) \\ &\equiv_p 2\,\mathrm{Re}\left(\frac{((-1)^{p'} + i)E_{p-3}}{2} - \frac{-q_2^2(1 - i(-1)^{p'}) + 4(-1)^{p'} E_{p-3}}{8}\right) \\ &\equiv_p \frac{q_2^2}{4}.\end{aligned}$$

□

Lemma 11. *We have*

$$\sum_{k=1}^{p'} \frac{(-1)^k}{k} H_{2\lfloor \frac{k}{2} \rfloor} \equiv_p \frac{q_2^2}{2}. \tag{16}$$

Proof. Since $H_{2k} \equiv_p H_{2(p'-k)}$,

$$\begin{aligned}
\sum_{k=1}^{p'} \frac{(-1)^k}{k} H_{2\lfloor \frac{k}{2} \rfloor} &= \sum_{k=1}^{p''} \frac{H_{2k}}{2k} - \sum_{k=1}^{\lceil \frac{p'}{2} \rceil} \frac{H_{2(k-1)}}{2k-1} \\
&= \sum_{k=1}^{p''} \frac{H_{2k}}{2k} - \sum_{k=1}^{p'} \frac{H_{2(p'-k)}}{p-2k} + \sum_{k=1}^{p''} \frac{H_{2(p'-k)}}{p-2k} \\
&\equiv_p \sum_{k=1}^{p''} \frac{H_{2k}}{2k} + \sum_{k=1}^{p'} \frac{H_{2k}}{2k} - \sum_{k=1}^{p''} \frac{H_{2k}}{2k} = \frac{1}{2} \sum_{k=1}^{p'} \frac{H_{2k}}{k} \equiv_p \frac{q_2^2}{2}
\end{aligned}$$

where in the last step we used (13). □

Lemma 12. *We have*

$$\sum_{k=1}^{p'} \frac{(-1)^k}{k} H_{\lfloor \frac{k}{2} \rfloor} \equiv_p q_2^2 + (-1)^{\frac{p-1}{2}} E_{p-3}. \tag{17}$$

Proof. The argument goes as follows:

$$\begin{aligned}
\sum_{k=1}^{p'} \frac{(-1)^k}{k} H_{\lfloor \frac{k}{2} \rfloor} &= \sum_{k=1}^{p''} \frac{H_k}{2k} - \sum_{k=1}^{\lceil \frac{p'}{2} \rceil} \frac{H_{k-1}}{2k-1} \\
&= \sum_{k=1}^{p''} \frac{H_k}{2k} - \sum_{k=1}^{p'} \frac{H_{p'-k}}{p-2k} + \sum_{k=1}^{p''} \frac{H_{p'-k}}{p-2k} \\
&\equiv_p \sum_{k=1}^{p''} \frac{H_k}{2k} + \sum_{k=1}^{p'} \frac{H_{p'-k}}{2k} - \sum_{k=1}^{p''} \frac{H_{p'-k}}{2k}.
\end{aligned}$$

By using

$$H_{p'-k} = H_{p'} - \sum_{j=0}^{k-1} \frac{1}{p'-j} \equiv_p H_{p'} + 2 \sum_{j=0}^{k-1} \frac{1}{2j+1} = H_{p'} + 2H_{2k} - H_k$$

we get

$$\sum_{k=1}^{p'} \frac{(-1)^k}{k} H_{\lfloor \frac{k}{2} \rfloor} \equiv_p -\frac{1}{2} \sum_{k=1}^{p'} \frac{H_k}{k} + \sum_{k=1}^{p''} \frac{H_k}{k} + \sum_{k=1}^{p'} \frac{H_{2k}}{k} - \sum_{k=1}^{p''} \frac{H_{2k}}{k} - \frac{H_{p'}^2}{2} + \frac{H_{p'} H_{p''}}{2}.$$

Invoke (2), (4), (7), (8), (14), and (15) to complete the proof. □

Finally, we are ready to state and prove our main result.

Theorem 1. *We have*

$$\sum_{k=1}^{p'} \frac{(-1)^k}{k} \sum_{j=\lfloor \frac{k}{2} \rfloor + 1}^{k} \frac{1}{2j-1} \equiv_p 0. \tag{18}$$

Proof. The inner sum can be rewritten as:

$$\sum_{\lfloor \frac{k}{2} \rfloor + 1}^{k} \frac{1}{2j-1} = \sum_{j=1}^{k} \frac{1}{2j-1} - \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{2j-1} = \left[H_{2k} - \frac{1}{2} H_k \right] - \left[H_{2\lfloor \frac{k}{2} \rfloor} - \frac{1}{2} H_{\lfloor \frac{k}{2} \rfloor} \right].$$

In view of this, the theorem can be reformulated as

$$\sum_{k=1}^{p'} \frac{(-1)^k}{k}\left[H_{2k} - \frac{1}{2}H_k - H_{2\lfloor\frac{k}{2}\rfloor} + \frac{1}{2}H_{\lfloor\frac{k}{2}\rfloor}\right] \equiv_p 0.$$

The proof follows from (12), (15), (16), and (17). □

4. An infinite series evaluation

In the present section, we consider an infinite series counterpart to the harmonic sum that has been the subject of much of this paper.

Theorem 2. *We have*

$$S := \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sum_{j=\lfloor\frac{k}{2}\rfloor+1}^{k} \frac{1}{2j-1} = -\frac{5}{8}\zeta(2).$$

Proof. Split the sum S according to parity to express in terms of harmonic sums,

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \frac{H_{4k} - \frac{1}{2}H_{2k} - H_{2k} + \frac{1}{2}H_k}{2k} - \sum_{k=1}^{\infty} \frac{H_{4k-2} - \frac{1}{2}H_{2k-1} - H_{2k-2} + \frac{1}{2}H_{k-1}}{2k-1} \\ &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k} H_{2k} - \frac{3}{2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k} H_k - \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{(-1)^k}{k} H_{\lfloor\frac{k}{2}\rfloor}. \end{aligned}$$

Next, we compute each series one-by-one, the easiest of which being $\sum_{k\geq 1} \frac{1}{(2k-1)^2} = \frac{3}{4}\zeta(2)$. For the rest, the representation $H_k = \int_0^1 \frac{1-t^k}{1-t}dt$ will be employed repeatedly.

$$\sum_{k\geq 1} \frac{(-1)^k H_k}{k} = \int_0^1 \frac{dt}{1-t} \sum_{k\geq 1} \frac{(-1)^k - (-t)^k}{k} = \int_0^1 \frac{\log(1+t) - \log 2}{1-t} dt = \frac{\log^2 2 - \zeta(2)}{2},$$

$$\begin{aligned} \sum_{k\geq 1} \frac{(-1)^k H_{2k}}{k} &= \int_0^1 \frac{\log(1+t^2) - \log 2}{1-t} dt = \int_0^1 \left[\frac{\log(1+t^2) - \log 2}{1-t^2}\right](1+t)\,dt \\ &= \int_0^1 \frac{\log(1+t^2) - \log 2}{1-t^2}\,dt + \frac{1}{2}\int_0^1 \frac{\log(1+t) - \log 2}{1-t}\,dt \\ &= -\frac{3}{8}\zeta(2) + \frac{\log^2 2 - \zeta(2)}{4}, \end{aligned}$$

$$\sum_{k\geq 1} \frac{(-1)^k}{k} H_{\lfloor\frac{k}{2}\rfloor} = \frac{1}{2}\int_0^1 \frac{\frac{1}{\sqrt{t}}\log\left(\frac{1+\sqrt{t}}{1-\sqrt{t}}\right) + \log(1-t) - 2\log 2}{1-t}\,dt = S_1 + S_2 + S_3;$$

where

$$\begin{aligned} S_1 &:= \frac{1}{2}\int_0^1 \frac{\log(1+\sqrt{t}) - \log 2}{1-t}\,dt = \int_0^1 \left(\frac{\log(1+t) - \log 2}{1-t^2}\right) t\,dt \\ &= \frac{1}{2}\int_0^1 \frac{\log(1+t) - \log 2}{1-t}\,dt - \frac{1}{2}\int_0^1 \frac{\log(1+t) - \log 2}{1+t}\,dt \\ &= \frac{\log^2 2 - \zeta(2)}{4} + \frac{1}{4}\log^2 2, \end{aligned}$$

$$S_2 := \frac{1}{2}\int_0^1 \log(1-\sqrt{t})\left[\frac{1-\frac{1}{\sqrt{t}}}{1-t}\right] dt = -\int_0^1 \frac{\log(1-t)}{1+t}\,dt = \frac{\zeta(2)-\log^2 2}{2},$$

$$\begin{aligned}S_3 &:= \frac{1}{2}\int_0^1 \frac{\frac{1}{\sqrt{t}}\log(1+\sqrt{t})-\log 2}{1-t}\,dt = \int_0^1 \frac{\log(1+t)-t\log 2}{1-t^2}\,dt \\ &= \frac{1}{2}\int_0^1 \frac{\log(1+t)-\log 2}{1-t}\,dt + \frac{1}{2}\int_0^1 \frac{\log(1+t)-\log 2}{1+t}\,dt + \log 2\int_0^1 \frac{dt}{1+t} \\ &= \frac{\log^2 2-\zeta(2)}{4} - \frac{1}{4}\log^2 2 + \log^2 2.\end{aligned}$$

Combining all the above calculations yields $S = -\frac{5}{8}\zeta(2)$, as required. □

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