On the p-adic valuations of values of Legendre polynomials

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Abstract

We prove an explicit formula for the p-adic valuation of the Legendre polynomials $P_n(x)$ evaluated at a prime p, and generalize an old conjecture of the third author. We also solve a problem proposed by Cigler in 2017.

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1 Introduction

The p-adic valuation is a key concept in number theory that quantifies the divisibility of a an integer n by a prime number p. It is denoted $\nu_p(n)$ and defined as the largest nonnegative integer k such that p^k divides n. By convention we set $\nu_p(0) = +\infty$. More generally, for a nonzero rational number m/n, we have $\nu_p(m/n) := \nu_p(m) - \nu_p(n)$. The p-adic valuation is useful for understanding the arithmetic properties of integers, with applications to Diophantine equations, congruences, the p-adic numbers, and local fields.

Determining the p-adic valuation of the elements of various combinatorial sequences is an old, interesting, and often challenging problem. For example, in 1830 Legendre [14, p. 10] gave a celebrated formula for $\nu_p(n!)$:

$$\nu_p(n!) = \sum_{i\geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor . \tag{1}$$

Although this sum is formally over infinitely many values of i, its terms are 0 for all sufficiently large i. An alternative formulation is

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1}, \tag{2}$$

where $s_p(n)$ denotes the sum of the base-p digits of n. Later, Kummer [13] gave a formula for $\nu_p(\binom{n}{k})$; namely, he expressed it as the number of carries in the base-p addition of k and $n-k$.

Since then, the p-adic valuations of many other sequences have been studied. Among them are the Fibonacci and tribonacci numbers studied by Lengyel [15] and by Marques and Lengyel [16], respectively.

In some instances, the p-adic valuation of a sequence $(c(n))_{n\geq 0}$ is *p-regular*, that is, the p-kernel of the sequence $(\nu_p(c(n)))_{n\geq 0}$ produces a finitely generated module [2, 4]. In other words, the set of subsequences

$$\{(\nu_p(c(p^e n + i)))_{n\geq 0} \colon e \geq 0, 0 \leq i < p^e\}$$

is of finite rank over the rationals. For example, the sequence $c(n) = \binom{2n}{n}$ is p-regular, since from Eq. (2) it follows that the p-kernel of $(\nu_p(c(n)))_{n\geq 0}$ is spanned by the three sequences $(\nu_p(c(n)))_{n\geq 0}$, $(\nu_p(c(pn + p - 1)))_{n\geq 0}$, and the constant sequence 1.

Boros, Moll, and the third author studied the 2-regularity of the 2-adic valuation of certain polynomials associated with definite integrals [8]. Bell [6] and Medina, Moll, and Rowland [17] studied the case of polynomial $c(n)$ more generally. To name a few other papers, Shu and Yao [26] characterized analytic functions $f\colon \mathbb{Z}_p \to \mathbb{C}_p$ without roots in $\mathbb{N}$ such that $(\nu_p(f(n)))_{n\geq 0}$ is p-regular. Medina and Rowland [18] further studied the p-regularity of the Fibonacci numbers, and Murru and Sanna [19] analyzed that of the more general Lucas sequences.

In other cases, the p-adic valuations exhibit various kinds of regularities without actually being p-regular. For example, see [1, 5, 11].

Legendre polynomials have a long history of being studied in number theory; we can mention, for example, their use in irrationality proofs [7] and in the Hasse invariant of certain elliptic curves [9]. Their irreducibility is the subject of a famous conjecture of Stieltjes [27]. Understanding the divisibility of their values by primes, therefore, could provide additional insight for these questions. In this paper, we study the p-adic valuations of the Legendre polynomials $P_n(x)$ evaluated at a prime number p, and show that they are p-regular. In the next section, we provide some additional motivation for studying this question.

2 Motivation

Back in 1988, when the third author (JS) was an assistant editor of the problems section of the *American Mathematical Monthly*, he received a submission from Nicholas Strauss and Derek Hacon with a proof of an inequality about the 3-adic valuation of the sequence $d(n) := \sum_{0 \leq i < n} \binom{2i}{i}$. The sequence $d(n)$ is present in the On-Line Encyclopedia of Integer Sequences (OEIS) [20] as sequence A006134.

JS guessed that $A(n) := (\nu_3(d(n)))$ might be a 3-regular sequence and used a computer program to discover the following heuristic relations:

$$
\begin{aligned}
A(3n+2) &= A(n) + 2 \\
A(9n) &= A(3n) \\
A(9n+1) &= A(3n) + 1 \\
A(9n+3) &= A(3n) \\
A(9n+4) &= A(3n+1) + 1 \\
A(9n+6) &= A(3n+1) \\
A(9n+7) &= A(3n+1) + 1.
\end{aligned}
$$

They led JS to conjecture that

$$\nu_3(d(n)) = \nu_3(\binom{2n}{n}) + 2\nu_3(n),$$

which he was later able to prove with the helpful advice of Jean-Paul Allouche. The original problem proposal, modified to give the exact formula, eventually appeared as Problem 6625 in the *American Mathematical Monthly* in 1990 [28], and a completely different solution by Don Zagier, based on 3-adic analysis, was published two years later [29]. The sequence $\nu_3(d(n))$ is the sequence A082490 in the OEIS.

After this, JS was inspired to use the same computer program to explore whether the p-adic valuations of other combinatorial sums might (conjecturally) have similar identities.

This effort was largely unsuccessful, with the exception of the following sequence: Define

$$a(n) := \sum_{0\le i\le n} \binom{n}{i}\binom{n+i}{i}, \tag{3}$$

known as *central Delannoy numbers* and listed as sequence A001850 in OEIS. Define $b(n) := \nu_3(a(n))$; this is sequence A358360 in the OEIS. Then the numerical evidence supported the following conjecture.

Conjecture 1. The sequence $(b(i))_{i\ge 0}$ satisfies the following identities:

$$b(i) = \begin{cases} b(\lfloor i/3\rfloor) + (\lfloor i/3\rfloor \bmod 2), & \text{if } i \equiv 0, 2 \pmod 3; \\ b(\lfloor i/9\rfloor) + 1, & \text{if } i \equiv 1 \pmod 3). \end{cases}$$

This conjecture appeared in [3, p. 453], and JS frequently mentioned it in his talks (e.g., [24]). This conjecture was proved only in 2023 by Shen [25].

In 2017, JS posted the conjecture and its generalization to arbitrary prime p as a query on the MathOverflow website [23], which ultimately inspired the authors to form a team and settle the generalized conjecture (Theorem 6 below) with a joint effort. Namely, we prove explicit and recurrence formulae for $\nu_p(P_n(p))$ for every prime $p \ge 3$, from which Conjecture 1 follows as a partial case with $p = 3$ and $a(n) = P_n(3)$.

As we will see in Theorem 8, our results also imply the identity $\nu_p(M_n(p)) = \nu_p(P_n(p))$ conjectured by Cigler [10] for the polynomials

$$M_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^k. \tag{4}$$

Along the lines of these results, we also pose the following open question.

Conjecture 2. For every integer $n \ge 0$, we have

$$\nu_3\Big(\sum_{k=0}^{n} \binom{n}{k}^3 2^k\Big) = \begin{cases} s_3(\frac{n-1}{2}) + 1, & \text{if } n \equiv -1 \pmod 6; \\ s_3(\lfloor \frac{n+1}{2} \rfloor), & \text{otherwise.} \end{cases}$$

3 Main results

In this section, we state the main results of the paper.

Theorem 3. *Let p be a prime number and r be a rational number such that $\nu_p(r) \ge 1$. Then for every integer $n \ge 0$, we have*

$$\nu_p(P_n(r)) = \begin{cases} \nu_p(\binom{n}{n/2}), & \text{if } n \text{ is even and } p \ge 3; \\ \nu_p(\binom{n-1}{(n-1)/2}) + \nu_p(r) + \nu_p(n), & \text{if } n \text{ is odd and } p \ge 3; \\ \nu_2(\binom{n}{n/2}) - n, & \text{if } n \text{ is even and } p = 2; \\ \nu_2(\binom{n-1}{(n-1)/2}) + \nu_2(r) + 1 - n, & \text{if } n \text{ is odd and } p = 2. \end{cases}$$

Equivalently,

$$\nu_p(P_n(r)) = \nu_p(\frac{1}{2^n}\binom{n}{\lfloor n/2\rfloor}) + (n \bmod 2)\nu_p(r(n+1)).$$

Theorem 4. *Let $p \geq 3$ be a prime number. Then for every integer $m \geq 0$, we have*

$$\nu_p(P_{2m}(p)) = \nu_p(\binom{2m}{m});$$

$$\nu_p(P_{2m+1}(p)) = 1 + \nu_p(2m+1) + \nu_p(\binom{2m}{m}).$$

Moreover, for every integer $n \geq 0$,

$$\nu_p(P_n(p)) = \frac{2s_p(\lfloor n/2\rfloor) - s_p(n) + (n \bmod 2)p}{p-1}.$$

Theorem 5. *For every integer $n \geq 0$, we have*

$$\nu_2(P_n(2)) = (n \bmod 2) - \nu_2(n!).$$

Theorem 6. *Let $p \geq 3$ be a prime number and $f(n) := \nu_p(P_n(p))$. Then for all integers $n \geq 0$ and $0 \leq a < p$, we have*

$$f(pn+a) = \begin{cases} f(n) + (n \bmod 2), & \text{if } a \text{ is even;} \\ f(n) + 1 - (n \bmod 2), & \text{if } a \text{ is odd.} \end{cases}$$

For the special case $p = 3$, with the aid of Eq. (5) for $x = 3$, Theorem 6 implies Conjecture 1 by setting $i = 3n + a$ with $0 \leq a < 3$. The case of $a \in \{0, 2\}$ is immediate, while for $a = 1$, we get $b(i) = b(n) + 1 - (n \bmod 2)$. The case $a = 1$ then follows by writing $n = 3m + b$ and noticing that $n \bmod 2 = m \bmod 2$ when $b \in \{0, 2\}$, while $(m \bmod 2) + (n \bmod 2) = 1$ when $b = 1$. We thus recover the recent result of Shen [25].

As a corollary of Theorems 5 and 6 we also get

Corollary 7. *For every prime number p, the sequence $(\nu_p(P_n(p)))_{n\geq 0}$ is p-regular.*

Theorem 8. *Let $M_n(x)$ be defined as in Eq. (4). Then for every integer $n \geq 0$ and prime $p \geq 3$, we have*

$$\nu_p(M_n(p)) = \nu_p(P_n(p)).$$

4 Background and preliminary results

There exist many formulae for Legendre polynomials $P_n(x)$, including the following identity [12]:

$$P_n(x) = \sum_{0\leq k\leq n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k, \tag{5}$$

which can also be expressed in the form of hypergeometric series [21, §93, p. 166, Eq. (2)]. The following two formulas for Legendre polynomials are due to Rodrigues [22]:

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}; \tag{6}$$

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^k (x+1)^{n-k}. \tag{7}$$

Let p be a prime. As before, let $s_p(n)$ denote the sum of the base-p digits of n. From Eq. (2), for integers $n \geq k \geq 0$ we easily obtain

$$\nu_p(\binom{n}{k}) = \nu_p(\frac{n!}{k!(n-k)!}) = \frac{s_p(k) + s_p(n-k) - s_p(n)}{p-1}. \tag{8}$$

Moreover, for $n \geq 0$ and $0 \leq a < p$, we have

$$s_p(np + a) = s_p(n) + a. \tag{9}$$

The following lemma, together with Eq. (8), will be a key to expressing $\nu_p(P_n(p))$ in terms of s_p.

Lemma 9. *For every prime p and every integer $m \geq 0$,*

$$\nu_p(2m+1) + \nu_p(\binom{2m}{m}) = \nu_p(m+1) + \nu_p(\binom{2m+1}{m}) = \frac{2s_p(m) - s_p(2m+1) + 1}{p-1}.$$

Proof. The first equality follows from the identity $(2m+1)\binom{2m}{m} = (m+1)\binom{2m+1}{m}$. To prove the second equality, we consider two cases depending on whether p divides $2m+1$.

When $\nu_p(2m+1) = 0$, we have $s_p(2m+1) = s_p(2m) + 1$ and use Eq. (8) to get

$$\nu_p(2m+1) + \nu_p(\binom{2m}{m}) = \frac{2s_p(m) - s_p(2m)}{p-1} = \frac{2s_p(m) - s_p(2m+1) + 1}{p-1}.$$

When $\nu_p(2m+1) \geq 1$, we have $\nu_p(m+1) = 0$ and thus $s_p(m+1) = s_p(m) + 1$, implying that

$$\nu_p(m+1) + \nu_p(\binom{2m+1}{m}) = \frac{s_p(m) + s_p(m+1) - s_p(2m+1)}{p-1} = \frac{2s_p(m) - s_p(2m+1) + 1}{p-1}.$$

□

In view of formula (6), let us define

$$Q_n(x) := 2^n P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}. \tag{10}$$

Lemma 10. *Let p be a prime number and r be a rational number such that $\nu_p(r) \geq 1$. Then for every integer $m \geq 0$, we have*

$$\nu_p(Q_{2m}(r)) = \nu_p(\binom{2m}{m}).$$

Proof. Since $Q_0(r) = 1$, the lemma statement is clearly true for $m = 0$. From now on, let $m \geq 1$. By the definition (10) of $Q_n(x)$, we have

$$Q_{2m}(x) = \sum_{i=0}^{2m} a_i x^i,$$

where

$$a_i := \begin{cases} (-1)^{m-i/2}\binom{2m}{m-i/2}\binom{2m+i}{2m}, & \text{if } i \text{ is even;} \\ 0, & \text{if } i \text{ is odd.} \end{cases}$$

In particular, we have $a_0 = (-1)^m\binom{2m}{m}$, and so

$$\nu_p(a_0) = \nu_p(\binom{2m}{m}). \tag{11}$$

Now our goal is to show

$$\nu_p(a_i r^i) > \nu_p(a_0) \quad \text{for all even } 2 \leq i \leq 2m. \tag{12}$$

Let $i = 2j$. Equivalently to (12), we need to show that

$$\nu_p(a_{2j}/a_0) \geq -2j \cdot \nu_p(r) + 1 \quad \text{for all } 1 \leq j \leq m.$$

Since $\nu_p(r) \geq 1$, it suffices to show that

$$\nu_p(a_{2j}/a_0) \geq -2j + 1 \quad \text{for all } 1 \leq j \leq m. \tag{13}$$

Expanding a_{2j}/a_0, and regrouping terms, we obtain

$$\begin{aligned} |a_{2j}/a_0| &= \binom{2m}{m-j}\binom{2m+2j}{2m} \cdot \binom{2m}{m}^{-1} \\ &= \frac{(2m)!}{(m-j)!(m+j)!} \cdot \frac{(2m+2j)!}{(2m)!(2j)!} \cdot \frac{m!m!}{(2m)!} \\ &= \frac{(2m+2j)!m!}{(2m)!(m+j)!} \cdot \frac{m!}{(m-j)!} \cdot \frac{1}{(2j)!} \\ &= \frac{\prod_{\ell=1}^{2j}(2m+\ell)}{\prod_{\ell=1}^{j}(m+\ell)} \cdot \frac{m!}{(m-j)!} \cdot \frac{1}{(2j)!}. \end{aligned}$$

It is easy to see that $\prod_{\ell=1}^{j}(m+\ell) = \frac{1}{2^j}\prod_{\ell=1}^{j}(2m+2\ell)$ divides $\prod_{\ell=1}^{2j}(2m+\ell)$, implying that $a_{2j}/a_0 = b/(2j)!$ for some non-zero integer b. By Legendre's formula (2) we have

$$\nu_p((2j)!) = \frac{2j - s_p(2j)}{p-1} \leq 2j-1.$$

Thus, $\nu_p(a_{2j}/a_0) = \nu_p(b) - \nu_p((2j)!) \geq -2j+1$, and so inequalities (13) and (12) hold, implying that $\nu_p(Q_{2m}(r)) = \nu_p(a_0)$, which together with Eq. (11) completes the proof. □

Lemma 11. *Let p be a prime number, $m \geq 0$ be an integer, and r be a rational number such that $\nu_p(r) \geq 1$. When $p \geq 3$, we have*

$$\nu_p(Q_{2m+1}(r)) = \nu_p(r) + \nu_p(2m+1) + \nu_p(\binom{2m}{m}),$$

and for $p = 2$,

$$\nu_2(Q_{2m+1}(r)) = 1 + \nu_2(r) + \nu_2(\binom{2m}{m}).$$

Proof. The proof idea is similar to that of Lemma 10. Since $Q_1(r) = 2r$, the statement of the lemma is clearly true for $m = 0$. From now on, let $m \geq 1$. By the definition (10), we have

$$Q_{2m+1}(x) = \sum_{i=0}^{2m+1} a_i x^i,$$

where

$$a_i = \begin{cases} 0, & \text{if } i \text{ is even;} \\ (-1)^{m-(i-1)/2}\binom{2m+1}{m-(i-1)/2}\binom{2m+1+i}{2m+1}, & \text{if } i \text{ is odd.} \end{cases}$$

In particular, we have $a_1 = 2(2m+1)\binom{2m}{m}$, and so

$$\nu_p(a_1 r) = \begin{cases} \nu_p(r) + \nu_p(2m+1) + \nu_p(\binom{2m}{m}), & \text{if } p \geq 3; \\ 1 + \nu_p(r) + \nu_p(\binom{2m}{m}), & \text{if } p = 2. \end{cases} \tag{14}$$

Now our goal is to show

$$\nu_p(a_i r^i) > \nu_p(a_1 r) \quad \text{for all odd } i,\ 3 \leq i \leq 2m+1. \tag{15}$$

Let $i = 2j+1$. Equivalently to (15), we need to show

$$\nu_p(a_{2j+1}/a_1) \geq -2j \cdot \nu_p(r) + 1 \quad \text{for all } j,\ 1 \leq j \leq m.$$

Since $\nu_p(r) \geq 1$, it suffices to show that

$$\nu_p(a_{2j+1}/a_1) \geq -2j + 1 \quad \text{for all } j,\ 1 \leq j \leq m. \tag{16}$$

Expanding a_{2j+1}/a_1, and regrouping terms, we obtain

$$\begin{aligned}|a_{2j+1}/a_1| &= \binom{2m+1}{m-j}\binom{2m+2j+2}{2m+1}\cdot\left(2(2m+1)\binom{2m}{m}\right)^{-1} \\ &= \frac{(2m+1)!}{(m-j)!(m+j+1)!}\cdot\frac{(2m+2j+2)!}{(2m+1)!(2j+1)!}\cdot\frac{1}{2(2m+1)}\cdot\frac{m!m!}{(2m)!} \\ &= \frac{(2m+2j+2)!m!}{2(2m+1)!(m+j+1)!}\cdot\frac{m!}{(m-j)!}\cdot\frac{1}{(2j+1)!} \\ &= \frac{\prod_{\ell=1}^{2j+2}(2m+\ell)}{2\prod_{\ell=1}^{j+1}(m+\ell)}\cdot\frac{m!}{(m-j)!}\cdot\frac{1}{(2j+1)!}.\end{aligned}$$

Since $2\prod_{\ell=1}^{j+1}(m+\ell) = \frac{1}{2^j}\prod_{\ell=1}^{j+1}(2m+2\ell)$ divides $\prod_{\ell=1}^{2j+2}(2m+\ell)$, it follows that $a_{2j+1}/a_1 = b/(2j+1)!$ for some non-zero integer b. By Legendre's formula (2), we have

$$\nu_p((2j+1)!) = \frac{2j+1-s_p(2j+1)}{p-1} \le 2j-1,$$

where we used that $s_p(2j+1)\ge 1$ and $p-1\ge 2$ when $p\ge 3$, and $s_2(2j+1)\ge 2$ when $p=2$. Thus, $\nu_p(a_{2j+1}/a_1) = \nu_p(b) - \nu_p((2j+1)!) \ge -2j+1$, and so inequalities (16) and (15) hold, implying that $\nu_p(Q_{2m+1}(r)) = \nu_p(a_1 r)$, which together with (14) completes the proof. □

5 Proofs of the main results

Theorem 3 follows directly from Lemmas 10 and 11. It further implies Theorem 4 by setting $r = p$ and using Lemma 9.

We now prove Theorem 5.

Proof of Theorem 5. Using Eqs. (8), (9), and (2) for $p = 2$ and $a = 0$, we have

$$\nu_2(\binom{2m}{m}) = 2s_2(m) - s_2(2m) = s_2(2m) = 2m - v_2((2m)!).$$

Combining this with Theorem 3 for $r = p = 2$, we have

$$\nu_2(P_n(2)) = \begin{cases} -v_2(n!) & \text{if } n \text{ is even;} \\ -v_2((n-1)!)+1 & \text{if } n \text{ is odd.}\end{cases}$$

It remains to note that for an odd n, $v_2((n-1)!) = v_2(n!)$, which then allows to combine the two cases as $\nu_2(P_n(2)) = (n \bmod 2) - v_2(n!)$. □

Next, we tackle Theorem 6.

Proof of Theorem 6. As the theorem statement stipulates, let $p \geq 3$ be a prime number and $f(n) := \nu_p(P_n(p))$. To evaluate $f(pn + a)$ for $n \geq 0$ and $0 \leq a < p$, we distinguish between four cases, according to the parities of n and a.

Case 1: both $n = 2m$ and $a = 2b$ are even. Using Theorem 4 and Eq. (9), we get

$$\begin{aligned} f(2(pm+b)) &= \frac{2s_p(pm+b) - s_p(2pm+2b)}{p-1} = \frac{2(s_p(m)+b) - (s_p(2m)+2b)}{p-1} \\ &= \frac{2s_p(m) - s_p(2m)}{p-1} = f(2m). \end{aligned} \tag{17}$$

Case 2: $n = 2m$ is even and $a = 2b+1$ is odd. Since $\nu_p(2pm+2b+1) = 0$, Theorem 4 and Eq. (17) yield

$$\begin{aligned} f(2(pm+b)+1) &= 1 + \nu_p(2pm+2b+1) + \nu_p(\binom{2(pm+b)}{pm+b}) \\ &= 1 + \nu_p(\binom{2(pm+b)}{pm+b}) = 1 + f(2(pm+b)) \\ &= 1 + f(2m). \end{aligned}$$

Case 3: $n = 2m+1$ is odd and $a = 2b$ is even. By Theorem 4 and Eq. (9), we have

$$\begin{aligned} f(p(2m+1)+2b) &= \frac{2s_p(pm+b+\frac{p-1}{2}) - s_p(p(2m+1)+2b) + p}{p-1} \\ &= 1 + \frac{2s_p(m) - s_p(2m+1) + p}{p-1} = 1 + f(2m+1). \end{aligned}$$

Case 4: both $n = 2m+1$ and $a = 2b+1$ are odd. Since $b < \frac{p-1}{2}$, by Theorem 4 and Eq. (9), we have

$$\begin{aligned} f(p(2m+1)+2b+1) &= \frac{2s_p(pm+b+\frac{p+1}{2}) - s_p(p(2m+1)+2b+1)}{p-1} \\ &= \frac{2s_p(m) - s_p(2m+1) + p}{p-1} = f(2m+1). \end{aligned}$$

□

Finally, we prove Theorem 8, thus resolving the question of Cigler.

Proof of Theorem 8. First, note that by substituting $x/(2-x)$ in Eq. (7) and comparing it with Eq. (4), we get

$$M_n(x) = (2-x)^n P_n(\tfrac{x}{2-x}). \tag{18}$$

For a prime $p \geq 3$, Eq. (18) implies

$$\nu_p(M_n(p)) = \nu_p(P_n(\tfrac{p}{2-p})).$$

Now set $r_1 = p$ and $r_2 = p/(p-2)$, and note that $\nu_p(r_1) = \nu_p(r_2) = 1$. Then by Theorem 3, $\nu_p(P_n(r_2)) = \nu_p(P_n(r_1))$, implying that $\nu_p(M_n(p)) = \nu_p(P_n(p))$. □

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