QUASIMODULAR FORMS ARISING FROM JACOBI'S THETA FUNCTION AND SPECIAL SYMMETRIC POLYNOMIALS

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ABSTRACT. Ramanujan derived a sequence of even weight $2n$ quasimodular forms $U_{2n}(q)$ from derivatives of Jacobi's weight $3/2$ theta function. Using the generating function for this sequence, one can construct sequences of quasimodular forms of all nonnegative integer weights with minimal input: a weight 1 modular form and a power series $F(X)$. Using the weight 1 form $\theta(q)^2$ and $F(X) = e^{X/2}$, we obtain a sequence $\{Y_n(q)\}$ of weight n quasimodular forms on $\Gamma_0(4)$ whose symmetric function avatars $\widetilde{Y}_n(\boldsymbol{x}^k)$ are the symmetric polynomials $T_n(\boldsymbol{x}^k)$ that arise naturally in the study of syzygies of numerical semigroups. With this information, we settle two conjectures about the $T_n(\boldsymbol{x}^k)$. Finally, we note that these polynomials are systematically given in terms of the Borel-Hirzebruch $\widehat{A}$-genus for spin manifolds, where one identifies power sum symmetric functions p_i with Pontryagin classes.

1. INTRODUCTION AND STATEMENT OF RESULTS

Using the classical weight $3/2$ Jacobi theta function [19, Thm. 1.60]

$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{k=0}^{\infty}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}} = 1 - 3q + 5q^3 - 7q^6 + \dots,$$

Ramanujan (in his "lost notebook" [20, p. 369]) defined the sequence of q-series

$$(1.1) \qquad U_{2n}(q) = \frac{1^{2n+1} - 3^{2n+1}q + 5^{2n+1}q^3 - 7^{2n+1}q^6 + \cdots}{1 - 3q + 5q^3 - 7q^6 + \cdots} = \frac{\sum_{k\geq 0}(-1)^k(2k+1)^{2n+1}q^{\frac{k(k+1)}{2}}}{\sum_{k\geq 0}(-1)^k(2k+1)q^{\frac{k(k+1)}{2}}}.$$

He observed that

$$U_0 = 1, \quad U_2 = E_2, \quad U_4 = \frac{1}{3}(5E_2^2 - 2E_4), \quad \text{and} \quad U_6 = \frac{1}{9}(35E_2^3 - 42E_2E_4 + 16E_6),$$

where E_2, E_4, and E_6 are the classical Eisenstein series

$$E_2(q) := 1 - 24\sum_{n=1}^{\infty}\sum_{d|n}dq^n, \quad E_4(q) := 1 + 240\sum_{n=1}^{\infty}\sum_{d|n}d^3q^n, \quad \text{and} \quad E_6(q) := 1 - 504\sum_{n=1}^{\infty}\sum_{d|n}d^5q^n.$$

Ramanujan also conjectured that each $U_{2n}(q)$ is a weight $2n$ quasimodular form on $\mathrm{SL}_2(\mathbb{Z})$, a claim confirmed by Berndt et al. [7, 8] in the early 2000s. Answering a question of Andrews and Berndt (see p. 364 of [5]), Singh and two of the authors recently found explicit formulas [1] for each $U_{2n}(q)$ as *traces of partition Eisenstein series* (see (1.8)).

The key to these formulas is the discovery of the generating function (see Th. 3.4 of [1])[a]

$$(1.2) \qquad \Omega(X) := \sum_{n=0}^{\infty}U_{2n}(q)\cdot\frac{X^{2n}}{(2n+1)!} = \frac{\sinh X}{X}\cdot\prod_{j\geq 1}\left[1 - \frac{4(\sinh^2 X)q^j}{(1-q^j)^2}\right].$$

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[a]In [1, Thm. 3.4] we have replaced sin with hyperbolic sinh to eliminate the $(-1)^t$ factor.

Since Ramanujan's forms all have even weight, this generating function becomes a device for producing infinite sequences of quasimodular forms of all nonnegative integer weights, where the coefficient of X^n has weight n. This method requires minimal additional input: a choice of a weight 1 modular form, a nonzero complex number α, and a formal power series

$$F(X) = \sum_{m=0}^{\infty} a(m)X^m.$$

We illustrate this method for the congruence subgroup $\Gamma_0(4)$, the elements of the ring (for example, see [16]) $\mathbb{C}[\theta, E_2, E_4, E_6] = \mathbb{C}[\theta, E_2, E_4, E_6, \dots]$, where $\theta(q)$ is the weight $1/2$ theta function

$$(1.3) \qquad \theta(q) = \prod_{k \geq 1}(1 - q^{2k})(1 + q^{2k-1})^2 = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots .$$

Using the weight 1 modular form $\theta(q)^2$, we obtain an infinite sequence of quasimodular forms, $Y_n(F, \alpha; q)$, one for every nonnegative integer weight n. Indeed, the coefficient of X^n of

$$(1.4) \qquad \sum_{n=0}^{\infty} Y_n(F, \alpha; q) \cdot \frac{X^n}{n!} := F(\theta(q)^2 \cdot X) \cdot \Omega(\alpha X)$$

is a weight n quasimodular form on $\Gamma_0(4)$. By letting $F(X) = e^{X/2}$ and $\alpha = 1/2$, we obtain

$$(1.5) \qquad \sum_{n \geq 0} Y_n(q) \cdot \frac{X^n}{n!} := \exp\left(\frac{\theta(q)^2 X}{2}\right) \cdot \Omega\left(\frac{X}{2}\right),$$

where for convenience, we let $Y_n(q) := Y_n\left(e^{X/2}, \frac{1}{2}; q\right)$.

Theorem 1.1. *For each non-negative integer n, the following are true.*
(1) We have that $Y_n(q) = \frac{1}{n+1} + O(q)$ is a weight n quasimodular form on $\Gamma_0(4)$.
(2) We have that

$$Y_n(q) = \frac{1}{2^n(n+1)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} \theta(q)^{2n-4k} U_{2k}(q).$$

Examples. The first few q-series take the form $Y_0(q) = 1$, and

$$Y_1(q) = \frac{1}{2}\theta^2 = \frac{1}{2} + 2q + 2q^2 + 2q^4 + 4q^5 + 2q^8 + 2q^9 + \dots,$$

$$Y_2(q) = \frac{3\theta^4 + E_2}{12} = \frac{1}{3} - 8q^4 - 24q^8 - 32q^{12} - 56q^{16} - 48q^{20} - \dots,$$

$$Y_3(q) = \frac{\theta^6 + \theta^2 E_2}{8} = \frac{1}{4} - q - 13q^2 - 40q^3 - 73q^4 - 122q^5 - \dots.$$

Each $Y_n(q)$ has a canonical symmetric function avatar, which we will relate to a symmetric polynomial arising from syzygies of numerical semigroups. To this end, we make use of recent research by two of the authors that linked [3, Th. 1.5] the sequence $\{U_{2n}(q)\}$ to the $\widehat{A}$-genus of spin manifolds, a special generating function of polynomials, discovered by Borel and Hirzebruch. This identification is not immediate because the $\widehat{A}$-genus is expressed in terms of polynomials in Pontryagin classes [15] and not as q-series. The connection involves special symmetric function avatars of the sequence $\{U_{2n}(q)\}$, a result that we will modify and then apply to the $\{Y_n(q)\}$ using Theorem 1.1 (2).

These identifications rely on partition Eisenstein series. For a partition $\lambda = (1^{m_1}, \dots, n^{m_n}) \vdash n$, where m_j is the multiplicity of j, the *partition Eisenstein series* is given by (see [1, (1.5)] or [2, (1.2)])

$$(1.6) \qquad \lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n}) \vdash n \qquad \longmapsto \qquad E_\lambda := E_2^{m_1} E_4^{m_2} \cdots E_{2n}^{m_n},$$

where the classical Eisenstein series (see Chapter 1 of [19]) are defined by

$$(1.7) \qquad E_{2k}(q) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n,$$

and B_{2k} is the $2k$th Bernoulli number and $\sigma_v(n) := \sum_{d|n} d^v$. If ϕ is a function on integer partitions and n is positive integer, then the *nth trace of partition Eisenstein series* [1, (1.6)] is defined by

$$\text{Tr}_n(\phi; q) := \sum_{\lambda \vdash n} \phi(\lambda) E_\lambda(q).$$

Singh and two of the authors [1, Th. 1.3 (1)] proved that each $U_{2n}(q)$ is the trace

$$(1.8) \qquad U_{2n}(q) = \text{Tr}_n(\phi_U; q),$$

where we let

$$\phi_U(\lambda) := (2n+1)! \, 4^n \cdot \prod_{j=1}^{n} \frac{1}{m_j!} \left(\frac{B_{2j}}{(2j)(2j)!} \right)^{m_j}.$$

Hence, Theorem 1.1 (2) can be reformulated in terms of traces of partition Eisenstein series.

Theorem 1.2. *If n is a nonnegative integer, then*

$$Y_n(q) = \frac{1}{2^n(n+1)} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} \theta(q)^{2n-4k} \, \text{Tr}_n(\phi_U; q).$$

Remark. We could have declared $\theta(q)^2$ as an Eisenstein series allowing us to reformulate the sums in Theorem 1.2 as partition Eisenstein traces over $\{\theta(q)^2, E_2, E_4, E_6, \dots\}$, along the lines of recent work by Bringmann, Pandey, and van Ittersum [9]. However, this would have required unnecessary new notation.

We use these identities to obtain avatars of the $U_{2n}(q)$. The correspondence between quasimodular forms and symmetric functions is made by substituting[b] each

$$E_{2k}(q) \longleftrightarrow p_{2k}(\boldsymbol{x}),$$

in (1.6) and (1.8. Here $p_{2k}(\boldsymbol{x}) = \sum_{i \geq 1} x_i^{2k}$ is the $2k$th *even* power sum symmetric function, where we let $\boldsymbol{x} = (x_1, x_2, \dots)$. In particular, we have

$$(1.9) \qquad \Psi(E_\lambda) = \prod_{i=1}^{n} \Psi(E_{2i})^{m_i} = p_2(\boldsymbol{x})^{m_1} p_4(\boldsymbol{x})^{m_2} \cdots p_{2n}(\boldsymbol{x})^{m_n},$$

which we extend linearly to define the symmetric function representation of $\Psi(U_{2n}(q))$ using (1.8).

In view of Theorem 1.2, to produce symmetric polynomial avatars of each $Y_n(q)$, we must also define $\Psi(\theta(q)^2)$. Although there are no weight 1 modular forms on $\text{SL}_2(\mathbb{Z})$, one can "think of" $\theta(q)^2$ as a proxy for $E_1(q)$ for several reasons. First, it has weight 1. Additionally, by setting $k = 1/2$ and replacing $\sigma_{2k-1}(n)$ with $\sigma_*(n) := d_1(n) - d_3(n)$, where $d_j(n)$ is the number of divisors of n congruent to j modulo 4, in (1.7), we find

$$\theta(q)^2 = 1 - \frac{2}{B_1} \sum_{n=1}^{\infty} \sigma_*(n)q^n = 1 + 4 \sum_{n=1}^{\infty} \sigma_*(n)q^n.$$

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[b]This identification is different from [3], where $p_k \longleftrightarrow E_{2k}$.

The only adjustment needed is a minor modification to the divisor function. Therefore, it is consistent to let $\Psi : \theta^2(q) \mapsto p_1(\boldsymbol{x}) = x_1 + x_2 + \dots$. Hence, thanks to Theorem 1.2, we define the symmetric polynomial counterpart to $Y_n(q)$ by

$$(1.10) \qquad \widetilde{Y}_n(\boldsymbol{x}) := \frac{1}{(n+1)2^n} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} p_1(\boldsymbol{x})^{n-2k} \cdot \Psi(U_{2k}(q)).$$

The first few examples are as follows:

$$\widetilde{Y}_0(\boldsymbol{x}) = 1, \quad \widetilde{Y}_1(\boldsymbol{x}) = \frac{1}{2}p_1(\boldsymbol{x}), \quad \widetilde{Y}_2(\boldsymbol{x}) = \frac{3p_1(\boldsymbol{x})^2 + p_2(\boldsymbol{x})}{12},$$

$$\widetilde{Y}_3(\boldsymbol{x}) = \frac{p_1(\boldsymbol{x})^3 + p_1(\boldsymbol{x})p_2(\boldsymbol{x})}{8}, \qquad \widetilde{Y}_4(\boldsymbol{x}) = \frac{15p_1(\boldsymbol{x})^4 + 30p_1(\boldsymbol{x})^2 p_2(\boldsymbol{x}) + 5p_2(\boldsymbol{x})^2 - 2p_4(\boldsymbol{x})}{240}.$$

In Section 2.1, we define symmetric functions $T_n(\boldsymbol{x}^k)$ (defined in (2.8) using polynomials $P_m(\boldsymbol{x}^k)$ given by (2.7)), where $\boldsymbol{x}^k = (x_1, x_2, \dots, x_k)$. These functions arise from a study of the alternating power sums of syzygies degrees of numerical semigroups. In [12], the second author explicitly computed many of these functions, where the first include the following.

$$T_0(\boldsymbol{x}^k) = 1, \qquad T_1(\boldsymbol{x}^k) = \frac{1}{2}p_1(\boldsymbol{x}^k), \qquad T_2(\boldsymbol{x}^k) = \frac{3p_1(\boldsymbol{x}^k)^2 + p_2(\boldsymbol{x}^k)}{12},$$

$$T_3(\boldsymbol{x}^k) = \frac{p_1(\boldsymbol{x}^k)^3 + p_1(\boldsymbol{x}^k)p_2(\boldsymbol{x}^k)}{8}, \quad T_4(\boldsymbol{x}^k) = \frac{15p_1(\boldsymbol{x}^k)^4 + 30p_1(\boldsymbol{x}^k)^2 p_2(\boldsymbol{x}^k) + 5p_2(\boldsymbol{x}^k)^2 - 2p_4(\boldsymbol{x}^k)}{240}.$$

One immediately notices the similarity between these symmetric functions. It is not an accident.

Theorem 1.3. *For every pair of positive integers k and n, we have $\widetilde{Y}_n(\boldsymbol{x}^k) = T_n(\boldsymbol{x}^k)$.*

Remark. The $\widetilde{Y}_n(\boldsymbol{x})$ are defined with infinitely many variables $\boldsymbol{x} = (x_1, x_2, \dots)$. In Theorem 1.3, we replace $\boldsymbol{x}$ by $\boldsymbol{x}^k = (x_1, \dots, x_k)$, which is required in the context arising from numerical semigroups.

Theorem 1.3, which stems from Ramanujan's $U_{2n}(q)$, unexpectedly offers deep insight into the theory of these symmetric polynomials. This knowledge allows us to solve two conjectures that the second author [12] formulated about $T_n(\boldsymbol{x}^k)$. The first is that the $T_n(\boldsymbol{x}^k)$ are modified versions of symmetric polynomials $f_n(\boldsymbol{x}^k)$ that arise from (restricted) integer partitions (see (2.11) and (2.12) for the definition and additional properties).

Conjecture 1.4. [Conjecture 2.1 of [12]] *If $n \geq 2$, then we have*

$$T_n(p_1, p_2, p_3, p_4, \dots, p_n) = f_n(p_1, -p_2, p_3, -p_4, \dots, p_n).$$

Theorem 1.5. *Conjecture 1.4 is true.*

The second conjecture of [12] claimed striking identities. To state the conjecture[c], we require the numbers A_{2j+1}, known as the *tangent/zig-zag numbers* (counting up/down permutations), which have the exponential generating function

$$\sec x + \tan x = \sum_{j \geq 0} A_j \cdot \frac{x^j}{j!}.$$

Conjecture 1.6. [Conjecture 3.1 of [12]] *For a positive integer n, we have*

$$\frac{T_{2n+1}(\boldsymbol{x}^k)}{T_1^{2n+1}(\boldsymbol{x}^k)} = \sum_{j=0}^{n} (-1)^j A_{2j+1} \binom{2n+1}{2j+1} \frac{T_{2n-2j}(\boldsymbol{x}^k)}{T_1^{2n-2j}(\boldsymbol{x}^k)}.$$

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[c]We note that we offer a more precise formulation of the conjecture [12, p. 62].

Theorem 1.7. *Conjecture 1.6 is true.*

These results illustrate the rich mathematics that arises from Ramanujan's $U_{2n}(q)$ by letting $F(X) = e^{X/2}$ and $\alpha = 1/2$ in (1.4). Hence, it is natural to ask the following open-ended questions.

Question 1. In view of Theorem 1.2 and [3, Th. 1.4], it is natural to ask whether the sequence $\{T_n(\boldsymbol{x}^k)\}$ encodes algebraic information about the Pontryagin classes [15] of spaces assembled from spin manifolds, perhaps through the framework of syzygies of the genera of numerical semigroups.

Question 2. Do other choices of weight 1 modular forms and power series $F(X)$ in (1.4) produce further important sequences of symmetric polynomials that are independent of the power sum symmetric functions $p_3, p_5, p_7, \ldots$?

Question 3. Does (1.4) generalize to other families of even weight quasimodular forms (i.e., replacing $U_{2n}(q)$ and the generating function $\Omega(X)$)? Is there a general theory, perhaps arising from the theory of Jacobi forms?

The proofs of these results make use of tools from umbral calculus, Pólya's cycle index theorem and involutions on symmetric functions. In Section 2, we recall the nuts and bolts that we require. Namely, in Subsection 2.1 we offer the required background on numerical semigroups and the symmetric polynomials $T_r(\boldsymbol{x}^k)$ and their algebraic properties, and in Subsection 2.2, we recall the symmetric functions $f_r(\mathbf{d}^k)$ that arise from restricted integer partitions. Finally, in Section 3 we compile these results to prove Theorems 1.1, 1.2, 1.3, 1.5, and 1.7.

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2. NUTS AND BOLTS

In this section, we recall basic facts about two families of symmetric polynomials, one arising from numerical semigroups, and the other arising from restricted integer partitions.

2.1. **Symmetric functions arising from syzygies of numerical semigroups.** Let $\mathbb{N}$ denote the set of all non-negative integers. A *numerical semigroup* $\langle d_1, \ldots, d_m \rangle$ is a subset $S_m \subset \mathbb{N}$ containing 0, closed under summation, with finite complement in $\mathbb{N}$. A set of generators $\{d_1, \ldots, d_m\}$ of a numerical semigroup S_m satisfies $\gcd(d_1, \ldots, d_m) = 1$ and is *minimal* if none of its proper subsets generates the numerical semigroup S_m [13]. Its generating function is given by

$$(2.1) \qquad H(S_m; z) = \sum_{s \in S_m} z^s, \qquad z < 1, \qquad 0 \in S_m,$$

and is known as *the Hilbert series* of S_m and has a rational representation (Rep),

$$(2.2) \qquad H(S_m; z) = \frac{Q(S_m; z)}{\prod_{i=1}^{m}(1 - z^{d_i})}, \qquad C_{k,j} \in \mathbb{N}, \ \ 1 \le k \le m-1, \ \ 1 \le j \le \beta_k,$$

where $\beta_0 = 1$, and

$$(2.3) \qquad Q(S_m; z) = 1 - \sum_{j=1}^{\beta_1} z^{C_{1,j}} + \sum_{j=1}^{\beta_2} z^{C_{2,j}} - \cdots \pm \sum_{j=1}^{\beta_{m-1}} z^{C_{m-1,j}}, \qquad \sum_{k=0}^{m-1}(-1)^k \beta_k = 0,$$

and $C_{k,j}$ and β_k stand for degrees of the kth syzygy and partial Betti's numbers, respectively [23].

We denote by $\mathbb{C}_k(S_m)$ the alternating power sum of syzygy degrees

$$(2.4) \qquad \mathbb{C}_k(S_m) := \sum_{j=1}^{\beta_1} C_{1,j}^k - \sum_{j=1}^{\beta_2} C_{2,j}^k + \ldots - (-1)^{m-1} \sum_{j=1}^{\beta_{m-1}} C_{m-1,j}^k.$$

We have that [10, Thm. 1]

$$\mathbb{C}_0(S_m) = 1, \qquad \mathbb{C}_r(S_m) = 0, \quad 1 \le r \le m-2, \qquad \mathbb{C}_{m-1}(S_m) = (-1)^m (m-1)! \cdot \pi_m,$$

where we let $\pi_m := \prod_{i=1}^m d_i$. For $r \ge m$, these sums were calculated and furnished in [11, (22)] as

$$(2.5) \qquad \mathbb{C}_n(S_m) = \frac{(-1)^m n!}{(n-m)!} \pi_m \cdot K_{n-m}(S_m), \qquad K_t(S_m) > 0, \ \ t \ge 0,$$

where $K_t(S_m)$ is a linear combination of the genera $G_0(S_m), \ldots, G_t(S_m)$ of the semigroup S_m, and $G_t(S_m) := \sum_{s \in \Delta_m} s^t$. Here, $\Delta_m := \mathbb{N} \setminus S_m$ and $G_0(S_m) = \#\Delta_m$ denote a set of gaps and a genus of S_m, respectively.

A special kind of numerical semigroups S_m, with the first Betti number $\beta_1 = m-1$, is called a symmetric *complete intersection* (CI), where the freedom to choose generators $\{d_1, \ldots, d_m\}$ of the semigroup S_m^{CI} becomes restricted due to the relations

$$\beta_k = \beta_{m-k-1}, \quad \beta_1 = m-1, \quad \beta_{m-1} = 1, \quad C_{k,j} + C_{m-k-1,j} = C_{m-1,1}, \quad \min\{d_1, \ldots, d_m\} \ge m.$$

The rational Rep of its Hilbert series reads

$$(2.6) \qquad H\left(S_m^{CI}; z\right) = \frac{\prod_{j=1}^{m-1}\left(1 - z^{e_j}\right)}{\prod_{i=1}^m \left(1 - z^{d_i}\right)}, \ \ e_j = c_j d_j, \quad c_j \in \mathbb{N}, \ c_j \ge 2, \ e_j \ge 2(m+1).$$

The tuple $\boldsymbol{e}^{m-1} := (e_1, \ldots, e_{m-1})$ presents the $m-1$ degrees of the first syzygy for symmetric CI semigroup S_m^{CI}.

In contrast to $Q\left(S_m; z\right)$ of (2.2), (2.3)), the numerator in (2.6) is the special symmetric polynomial $P_n(\mathbf{e}^{m-1})$ introduced by the second author [12]:

$$(2.7) \qquad P_n(\boldsymbol{x}^m) := \sum_{j=1}^m x_j^n - \sum_{1 \le j < r}^m (x_j + x_r)^n + \sum_{1 \le j < r < i}^m (x_j + x_r + x_i)^n - \ldots - (-1)^m \left(\sum_{j=1}^m x_j\right)^n.$$

This polynomial allows [12] for the construction of the polynomials $T_n(\boldsymbol{x}^m)$ from

$$(2.8) \qquad P_n(\boldsymbol{x}^m) = \frac{(-1)^{m+1} n!}{(n-m)!} \chi_m T_{n-m}\left(\boldsymbol{x}^m\right), \qquad \chi_m = \prod_{j=1}^m x_j, \quad T_0\left(\boldsymbol{x}^m\right) = 1,$$

These polynomials enjoy numerous properties, such as the following.

Proposition 2.1. [12, Lm. 1.3] *We have the inequality* $T_n(x_1, \ldots, x_m) \ge 0$ *whenever* $x_1, \ldots, x_m \ge 0$.

Remark. From (2.8), one might not expect that these symmetric functions are independent of $p_{2j-1}(\boldsymbol{x}^k)$, where $j > 1$, which is a consequence of Theorem 1.3.

For completeness, we remind the reader on the importance of these polynomials $T_n(\boldsymbol{x}^m)$. They are conjectured (see Conjecture 2.2 below) to play a central role in the explicit description of alternating power sums of syzygy degrees for the genera of the semigroups S_m.

Conjecture 2.2. *If we let* $\sigma_j = \sum_{i=1}^m d_i^j$ *and* $\delta_j = \frac{\sigma_j - 1}{2^j}$, *then we have*

$$(2.9) \qquad K_t(S_m) = \sum_{r=0}^t \binom{t}{r} T_{t-r}(\sigma) G_r(S_m) + \frac{2^{t+1}}{t+1} T_{t+1}(\delta),$$

where $T_r(X) := T_r(X_1, \ldots, X_r)$ stand for symmetric polynomials in the power sums

$$(2.10) \qquad X_k(\boldsymbol{x}^m) := \sum_{j=1}^{m} x_j^k \qquad \text{for an } m\text{-tuple } \boldsymbol{x}^m := (x_1, \ldots, x_m).$$

2.2. Symmetric functions arising from partitions.

Polynomials $T_n(\boldsymbol{x}^k)$ are analogous to another sequence of polynomials $\{f_n\}$ that arise in the theory of integer partitions. Given $\mathbf{d}^k := \{d_1, \ldots, d_k\}$, consider the restricted partition function $W(s; \mathbf{d}^k)$ (see [4]), which counts integer partitions of $s \geq 0$ into k positive integers $(d_1, \ldots, d_k)$.

It is well-known (see, for example [10, Section 3.1]) that $W(s; \boldsymbol{d}^k)$ can be described in terms of finitely many quasipolynomials $W_q(s; \boldsymbol{d}^k)$, each containing a single q-periodic function with $W_1(s; \boldsymbol{d}^k)$ being a polynomial, given in the form

$$W(s; \boldsymbol{d}^k) = \sum_{\substack{q | d_i \\ 1 \leq i \leq k}} W_q(s; \boldsymbol{d}^k).$$

The $W_q(s; \boldsymbol{d}^k)$'s are called the *Sylvester waves*. In particular, $W_1(s; \boldsymbol{d}^k)$ is referred to as the first Sylvester wave and has fascinating properties. For example, $W_1(s; \boldsymbol{d}^k)$ is the polynomial part of $W(s; \boldsymbol{d}^k)$ having an explicit formula [10, (3.16) and (7.1)] given by

$$(2.11) \qquad W_1(s; \mathbf{d}^k) = \frac{1}{(k-1)! \pi_k} \sum_{j=0}^{k-1} \binom{k-1}{j} f_j(\mathbf{d}^k) s^{k-1-j}, \qquad f_j(\mathbf{d}^k) = \left(\sigma_1 + \sum_{i=1}^{k} \mathcal{B}_i d_i \right)^j,$$

where $\sigma_1 := \sum_{j=1}^{k} d_j$ and $\pi_k := \prod_{i=1}^{k} d_i$. For our purpose in this paper, we will replace $\boldsymbol{d}^k = (d_1, \ldots, d_k)$ by $\boldsymbol{x}^k = (x_1, \ldots, x_k)$ and σ_j by the power sum p_j in the variables $(x_1, \ldots, x_k)$. By treating the $\mathcal{B}_i$'s as independent random variables, the Bernoulli symbolism $(\mathcal{B}_i x_i)^n$ is understood as $\mathcal{B}_i^n x_i = B_i^n x_i^n$ where

$$\frac{z}{e^z - 1} = \sum_{n \geq 0} B_n \cdot \frac{z^n}{n!}$$

generates the Bernoulli numbers.

Examples. The first few symmetric polynomials

$$(2.12) \qquad f_n(\boldsymbol{x}^k) := \left(p_1 + \sum_{i=1}^{k} \mathcal{B}_i x_i \right)^n = f_n(p_1, \ldots, p_n),$$

expanded in power sums (for example, see [10, (7.2)]), are:

$$f_0(\boldsymbol{x}^k) = 1, \qquad f_1(\boldsymbol{x}^k) = \frac{1}{2} p_1, \qquad f_2(\boldsymbol{x}^k) = \frac{3p_1^2 - p_2}{12}, \qquad f_3(\boldsymbol{x}^k) = \frac{p_1^3 - p_1 p_2}{8},$$

$$f_4(\boldsymbol{x}^k) = \frac{15p_1^4 - 30p_1^2 p_2 + 5p_2^2 + 2p_4}{240}, \qquad f_5(\boldsymbol{x}^k) = \frac{3p_1^5 - 10p_1^3 p_2 + 5p_1 p_2^2 + 2p_1 p_4}{96},$$

$$f_6(\boldsymbol{x}^k) = \frac{63p_1^6 - 315p_1^4 p_2 + 315p_1^2 p_2^2 + 126p_1^2 p_4 - 35p_2^3 - 42p_2 p_4 - 16p_6}{4032}.$$

Remark. If we set all of the variables to x, then we obtain the *convolution Stirling polynomials* studied by Knuth [17]

$$f_1(x) = \frac{x}{2}, \quad f_2(x, x) = \frac{3x^2 - x}{12}, \quad f_3(x, x, x) = \frac{x^3 - x^2}{8}, \quad f_4(x, \ldots, x) = \frac{15x^4 - 30x^3 + 5x^2 + 2x}{240} \cdots,$$

which has generating function

$$\left(\frac{ze^z}{e^z - 1}\right)^x = \sum_{n=0}^{\infty} f_n(x, \ldots, x) \cdot \frac{z^n}{n!}.$$

3. Proofs of Theorems 1.1, 1.2, 1.3, 1.5, and 1.7

Here we prove the results from the Introduction using the material in the previous section.

3.1. **Proof of Theorems 1.1 and 1.2.** We consider (1.4) for a general power series

$$F(X) = \sum_{m=0}^{\infty} a(m)X^m.$$

Since we have

$$\Omega(\alpha X) = \sum_{t=0}^{\infty} U_{2t}(q) \cdot \frac{\alpha^{2t}X^{2t}}{(2t+1)!},$$

it follows that

$$Y_n(F, \alpha; q) = \sum_{m+2t=n} \frac{a(m) \cdot n! \alpha^{2t}}{(2t+1)!} \cdot \theta(q)^{2m} U_{2t}(q).$$

Since $\theta(q)$ is a weight $1/2$ modular form on $\Gamma_0(4) \subseteq \mathrm{SL}_2(\mathbb{Z})$ (for example, see Chapter 1 of [19]) and $U_{2t}(q)$ is a weight $2t$ quasimodular form on $\mathrm{SL}_2(\mathbb{Z})$, it follows that each $\theta(q)^{2m}U_{2t}(q)$ is a weight $m + 2t = n$ quasimodular form on $\Gamma_0(4)$, which in turn implies that $Y_n(F, \alpha; q)$ is as well.

The proof of Theorem 1.1 (1-2) follows easily from the Taylor expansion of $F(X) = e^{X/2}$ and direct algebraic manipulation. To obtain Theorem 1.2, one simply substitutes (1.8) (see also [1, Thm. 1.3 (1)]) into the formulas in Theorem 1.1 (2).

3.2. **Proof of Theorem 1.5.** We require a few elementary facts from probability theory in the context of umbral calculus (for background on umbral calculus, see [21]).

Lemma 3.1. *If we let $\mathcal{B}^n = B_n$ (Bernoulli numbers), $\mathcal{C}_n := \frac{(-1)^n}{n+1}$ and $\mathcal{D}_n := \frac{1}{n+1}$, then for positive integers n in umbral formalism (treating $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ as random variables) the following are true:*

(1) $(\mathcal{B} + \mathcal{D})^n = 0$.
(2) $(\mathcal{B} + \mathcal{C})^n = (-1)^n$.
(3) $1 + \mathcal{C} = \mathcal{D}$.
(4) $1 + \mathcal{B} = -\mathcal{B}$.
(5) $(-\mathcal{B})^n = \mathcal{B}^n$ for $n \neq 1$ with with $B_1 = -\frac{1}{2}$.

Proof. Since these claims are straightforward, we only prove two of them. To prove (4), we recall the generating function

$$e^{\mathcal{B}z} = \sum_{n\geq 0} \mathcal{B}^n \cdot \frac{z^n}{n!} = \sum_{n\geq 0} B_n \cdot \frac{z^n}{n!} = \frac{z}{e^z - 1}.$$

This implies that

$$e^{(1+\mathcal{B})z} = \frac{ze^z}{e^z - 1} = \frac{-z}{e^{-z} - 1} = e^{(-\mathcal{B})z}.$$

Similarly, we have that

$$e^{\mathcal{D}z} = \sum_{n\geq 0} \mathcal{D}^n \frac{z^n}{n!} = \sum_{n\geq 0} \frac{1}{n+1} \frac{z^n}{n!} = \frac{e^z - 1}{z}.$$

Therefore, (1) follows from the identity

$$e^{(\mathcal{B}+\mathcal{D})z} = \frac{z}{e^z - 1} \cdot \frac{e^z - 1}{z} = 1 = e^{0z},$$

which means that $\mathcal{B}$ and $\mathcal{D}$ annihilate each other. □

Using this lemma, we provide an umbral expression for $T_n(\boldsymbol{x}^k)$ in the spirit of (2.12).

Lemma 3.2. *If* $\mathcal{C}_i = \frac{(-1)^i}{i+1}$ *is a sequence and* $k, n \in \mathbb{N}$, *then we have*

$$T_n(\boldsymbol{x}^k) = \left(p_1 + \sum_{i=1}^{k} \mathcal{C}_i x_i \right)^n.$$

Proof. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, let $m_\lambda = m_{\lambda_1} m_{\lambda_2} \cdots$ denote the associated *monomial symmetric function* [22, Chapter 7] by m_λ defined by

$$m_\lambda := \sum_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

where the sum ranges over all *distinct* permutations $\alpha = (\alpha_1, \alpha_2, \dots)$ of the entries of λ. For example, $m_{(1,1)} = \sum_{i<j} x_i x_j$ and $m_{(2,1,1)} = \sum x_i^2 x_j x_k$ with i, j, k distinct.

Given a sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$, we write $\mathcal{U}_\lambda = \mathcal{U}_{\lambda_1} \mathcal{U}_{\lambda_2} \cdots$. From Lemma 3.1 (3), we obtain

$$\left(p_1 + \sum_i \mathcal{C}_i x_i \right)^n = \left(\sum_i \mathcal{D}_i x_i \right)^n.$$

Then, the multinomial theorem and umbral calculus together gives

$$(3.1) \qquad \left(\sum_{i=1}^{k} \mathcal{D}_i x_i \right)^n = \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \lambda_2, \dots} \mathcal{D}_\lambda m_\lambda.$$

We apply (3.1) with $\mathcal{D}$ replaced by $\mathbf{1} = (1, 1, \dots)$ and n replaced by $n' := n + k$, while we rank partitions according to their *lengths*, to reformulate (2.7) to obtain

$$
\begin{aligned}
\mathcal{P}_{n'}(\boldsymbol{x}^k) &= \sum_{t=1}^{k} (-1)^{t-1} \sum_{1 \le j_1 < \cdots < j_t \le n'} (x_{j_1} + \cdots + x_{j_t})^{n'} \\
&= \sum_{t=1}^{k} (-1)^{t-1} \sum_{1 \le j_1 < \cdots < j_t \le n'} \sum_{\ell=1}^{t} \sum_{\substack{1 \le s_1 < \cdots < s_\ell \le n' \\ s_1, \dots, s_\ell \in \{j_1, \dots, j_t\}}} \sum_{\substack{\mu \vdash n' \\ \ell(\mu) = \ell}} \binom{n}{\mu_1, \dots, \mu_\ell} m_\mu(x_{s_1}, \dots, x_{s_\ell}) \\
&= \sum_{\ell=1}^{k} \sum_{t=\ell}^{k} (-1)^{t-1} \sum_{1 \le j_1 < \cdots < j_t \le n'} \sum_{\substack{1 \le s_1 < \cdots < s_\ell \le n' \\ s_1, \dots, s_\ell \in \{j_1, \dots, j_t\}}} \sum_{\substack{\mu \vdash n' \\ \ell(\mu) = \ell}} \binom{n'}{\mu_1, \dots, \mu_\ell} m_\mu(x_{s_1}, \dots, x_{s_\ell}) \\
&= \sum_{\ell=1}^{k} \sum_{t=\ell}^{k} (-1)^{t-1} \sum_{1 \le j_1 < \cdots < j_t \le n'} \sum_{\substack{\mu \vdash n' \\ \ell(\mu) = \ell}} \binom{n'}{\mu_1, \dots, \mu_\ell} \left\{ \sum_{\substack{1 \le s_1 < \cdots < s_\ell \le n' \\ s_1, \dots, s_\ell \in \{j_1, \dots, j_t\}}} m_\mu(x_{s_1}, \dots, x_{s_\ell}) \right\}.
\end{aligned}
$$

For a given t, ranging in $\{1, \ldots, k\}$, counting the subsets $\{j_1, \ldots, j_t\}$ that contain $\{s_1, \ldots, s_\ell\}$ leads to

$$
\begin{aligned}
\mathcal{P}_{n'}(\boldsymbol{x}^k) &= \sum_{\ell=1}^{k} \sum_{\substack{\mu \vdash n' \\ \ell(\mu)=\ell}} \binom{n'}{\mu_1, \ldots, \mu_\ell} \sum_{1 \le s_1 < \cdots < s_\ell \le n'} m_\mu(x_{s_1}, \ldots, x_{s_\ell}) \sum_{t=\ell}^{k} (-1)^{t-1} \binom{k-\ell}{t-\ell} \\
&= (-1)^{k-1} \sum_{\substack{\mu \vdash n+k \\ \mu_i \ge 1}} \binom{n+k}{\mu_1, \ldots, \mu_k} m_\mu(x_1, \ldots, x_k) \\
&= (-1)^{k-1} \frac{(n+k)!}{n!} x_1 \cdots x_k \sum_{\lambda \vdash n} \binom{n}{\lambda_1, \cdots, \lambda_k} \frac{m_\lambda(x_1, \ldots, x_k)}{(\lambda_1+1) \cdots (\lambda_k+1)} \\
&= \frac{(-1)^{k-1}(n+k)! \cdot \prod_{j=1}^{k} x_j}{n!} \left(\sum_{i=1}^{k} \mathcal{D}_i x_i \right)^n,
\end{aligned}
$$

where the last equality is due to (3.1). The claim follows by applying (2.8). $\qquad\square$

The proof of Theorem 1.5 will also require the following fact about products of formal power series and the (unique) involution ω on symmetric functions, that is determined by the action $\omega(p_n) = (-1)^{n-1} p_n$ on power sum functions (see, for example [22, Chapter 7]).

Lemma 3.3. *If $f(z)$ is a formal power series with constant term 1 and $g(z) := \frac{1}{f(-z)}$, then*

$$
\omega \left(\prod_i f(zx_i) \right) = \prod_i g(zx_i).
$$

Proof. We begin the proof by letting

$$
L(z) = \log f(z) = \sum_{k \ge 1} L_k z^k,
$$

which in turns give $\log g(z) = -L(-z)$. Therefore, we find that

$$
\log \left(\prod_i f(zx_i) \right) = \sum_i \log f(zx_i) = \sum_i \sum_k L_k x_i^k z^k = \sum_k L_k p_k z^k,
$$

where p_k is the kth power sum symmetric function. Applying ω, we obtain

$$
\omega \left(\log \left(\prod_i f(zx_i) \right) \right) = \sum_k (-1)^{k-1} L_k p_k z^k = -\sum_i L(-zx_i) = \sum_i \log g(zx_i).
$$

The claimed product formula follows by exponentiation. $\qquad\square$

Proof of Theorem 1.5. Thanks to Lemma 3.1 (3) and (4), respectively, we have

$$
\left(p_1 + \sum_i \mathcal{C}_i x_i \right)^r = \left(\sum_i \mathcal{D}_i x_i \right)^r \qquad \text{and} \qquad \left(p_1 + \sum_i \mathcal{B}_i x_i \right)^r = \left(\sum_i (-\mathcal{B}_i) x_i \right)^r.
$$

By expressing these expressions in terms of the power sum bases for symmetric functions (utilizing the two umbral representations (2.12) and Lemma 3.2), we obtain

$$\left(\sum_i \mathcal{D}_i x_i\right)^r = T_r(p_1, p_2, \dots) = \sum_{\lambda \vdash r} g^\lambda p_\lambda(\boldsymbol{x}),$$

$$\left(\sum_i \widetilde{\mathcal{B}}_i x_i\right)^r = f_r(p_1, p_2, \dots) = \sum_{\lambda \vdash r} \widetilde{g}^\lambda p_\lambda(\boldsymbol{x}),$$

for some coefficients g^λ and $\widetilde{g}^\lambda$ where $\boldsymbol{x} = (x_1, x_2, \dots)$ and $\widetilde{\mathcal{B}}_i = -\mathcal{B}_i$.

Next, one employs Lemma 3.3 with the exponential generating functions

$$f(z) = \frac{e^z - 1}{z} \qquad \text{and} \qquad g(z) = \frac{z}{1 - e^{-z}},$$

respectively, for $\mathcal{D}_k = \frac{1}{k+1}$ and $(-1)^k B_k$. By extracting the degree r terms (in z) and noting that $\omega(p_k) = (-1)^{k-1} p_k$, we obtain the desired outcome that

$$T_r(p_1, p_2, p_3, p_4, \dots) = f_r(p_1, -p_2, p_3, -p_4, \dots).$$

$$\square$$

3.3. **Proof of Theorem 1.3.** Recalling (1.8) and the identification $\Psi(E_{2n}(q)) = p_{2n}(\boldsymbol{x}^k)$, we define

$$u_{2n}(\boldsymbol{x}^k) := \Psi(U_{2n}(q))$$

and

(3.2) $$\widetilde{u}_{2n}(\boldsymbol{x}^k) := \sum_{\lambda \vdash n} \phi_U(\lambda) \prod_{j=1}^n \Psi(-E_{2j}^{m_j}) \qquad \text{whenever } \lambda = (1^{m_1}, 2^{m_2}, \dots, n^{m_n}).$$

Example. We list the first few of the above two sequences of polynomials:

$$u_0 = 1, \quad u_2 = p_2, \quad u_4 = \frac{5p_2^2 - p_4}{3}, \quad u_6 = \frac{35p_2^3 - 42p_2p_4 + 16p_6}{9}.$$

$$\widetilde{u}_0 = 1, \quad \widetilde{u}_2 = -p_2, \quad \widetilde{u}_4 = \frac{5p_2^2 + p_4}{3}, \quad \widetilde{u}_6 = -\frac{35p_2^3 + 42p_2p_4 + 16p_6}{9}.$$

Remark. In this section, we make important use of Pólya's cycle index theorem (PCIT) [22, Section 5]

$$\exp\left(\sum_{j \geq 1} Z_j \frac{w^j}{j}\right) = \sum_{n \geq 0} \left(\sum_{\lambda \vdash n} \prod_{j=1}^n \frac{1}{m_j!} \left(\frac{Z_j}{j}\right)^{m_j}\right) w^n,$$

where we require writing a partition in the *frequency notation* $\lambda = (1^{m_1} \dots t^{m_n}) \vdash n$. PCIT is deeply intertwined with other key results and methods, offering a consistent approach to a variety of counting problems that are similar and adjacent to the results in this paper. Specifically, it provides a method for deriving generating functions for structures related to generalized Bernoulli and Bell polynomials [18], Bernoulli-Barnes polynomials [6] (with interesting connection to Fourier Dedekind sums), and it forms a crucial part of the standard exponential generating functions due to Touchard. [22, Eq'n. (5.30)].

We proceed with some relevant preparatory results. The first of which, Lemma 3.4), is an explicit and compact umbral representations of u_{2n} and $\widetilde{u}_{2n}$ (hence of U_{2n}, by association) in the form of

$$u_{2n} = (2n+1)\left(\sum_{i=1}^k (1 + 2\mathcal{C}_i)x_i\right)^{2n} \qquad \text{and} \qquad \widetilde{u}_{2n} = (2n+1)\left(\sum_{i=1}^k (1 + 2\mathcal{B}_i)x_i\right)^{2n},$$

where $\mathcal{C}$ and $\mathcal{B}$ are defined in Lemma 3.1. In this context, observe our use of the umbral mechanism for a k-tuple of symbol:

$$\mathcal{B}_1^{a_1}\mathcal{B}_2^{a_2}\cdots\mathcal{B}_k^{a_k} \longrightarrow B_{a_1}B_{a_2}\cdots B_{a_k},$$

where $a_1 + a_2 + \cdots + a_k = n$.

Lemma 3.4. *For each nonnegative integer n, we have*

$$\left(\sum_{i=1}^{k}(1+2\mathcal{C}_i)x_i\right)^n = \begin{cases} \frac{u_n}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

$$\left(\sum_{i=1}^{k}(1+2\mathcal{B}_i)x_i\right)^n = \begin{cases} \frac{\tilde{u}_n}{n+1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We only prove the second assertion, as the first one follows a similarly. To this end, we recall the series expansion [14, 1.518.1] for the reciprocal of what we call here the *sinhc function* $\text{sinhc}(t) = \frac{\sinh(t)}{t}$ (reminiscent of the *sinc function*) given by

$$(3.3) \qquad \frac{1}{\text{sinhc}(t)} = \exp\left(-\sum_{n\geq 1}\frac{4^n B_{2n}}{(2n)(2n)!}\cdot t^{2n}\right).$$

On the other hand, the (usual) exponential generating function for the Bernoulli numbers takes the form

$$\exp(\mathcal{B}z) = \sum_{n\geq 0}\mathcal{B}^n\frac{z^n}{n!} = \sum_{n\geq 0}B_n\frac{z^n}{n!} = \frac{z}{e^z - 1}$$

from which it is natural to deduce that

$$(3.4) \qquad \exp((1+2\mathcal{B}_i)x_iz) = \sum_{n\geq 0}((1+2\mathcal{B}_i)x_i)^n\frac{z^n}{n!} = \frac{2x_iz\cdot e^{x_iz}}{e^{2x_iz}-1} = \frac{1}{\text{sinhc}(x_iz)}.$$

Consequently, (3.4) implies the finite product

$$(3.5) \qquad \prod_{i=1}^{k}\frac{1}{\text{sinhc}(x_iz)} = \exp\left(z\sum_{i=1}^{k}(1+2\mathcal{B}_i)x_i\right) = \sum_{n\geq 0}\frac{z^n}{n!}\left(\sum_{i=1}^{k}(1+2\mathcal{B}_i)x_i\right)^n.$$

Combining (3.3), (3.5) and Pólya's cycle index theorem [22, Section 5], we arrive at

$$\sum_{n\geq 0}\frac{z^n}{n!}\left(\sum_{i=1}^{k}(1+2\mathcal{B}_i)x_i\right)^n = \prod_{i=1}^{k}\frac{1}{\text{sinhc}(x_iz)} = \exp\left(-\sum_{n\geq 1}\frac{4^n B_{2n}\, p_{2n}}{(2n)(2n)!}\cdot z^{2n}\right)$$

$$= \sum_{n\geq 0}z^{2n}\sum_{\lambda\vdash n}\left(\prod_{j=1}^{n}\frac{1}{m_j!}\left(\frac{-4^j B_{2j}}{(2j)(2j)!}\right)^{m_j}\right)\Psi(E_\lambda).$$

Compare the coefficients of z^n on both sides, together with (3.2), to complete the proof. $\qquad\square$

In [12, (25)]), one of the authors proved the identity

$$(3.6) \qquad f_{2n+1} = \sum_{j=1}^{2n+1}(-1)^{j+1}\binom{2n+1}{j}f_1^j f_{2n+1-j}.$$

The next result reproves (3.6) and extends it to all f_n.

Lemma 3.5. *For each positive integer n, we have the recursive formula that*

$$f_n = \begin{cases} \sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}f_1^j f_{n-j} & \text{if } n \text{ is odd,} \\ \sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}f_1^j f_{n-j} + \frac{\widetilde{u}_n}{(n+1)2^n} & \text{if } n \text{ is even.} \end{cases}$$

Proof. The use of umbral symbolism allows for the more compact reformulation

(3.7)
$$(f - f_1)^n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{\widetilde{u}_n}{(n+1)2^n} & \text{if } n \text{ is even.} \end{cases}$$

It suffices to only consider the case n is even. Recalling (2.12), expand the left-hand side as

$$(f - f_1)^{2n} = \left(p_1 + \sum_{i=1}^{k}\mathcal{B}_i x_i - \frac{1}{2}p_1 \right)^{2n} = \frac{1}{4^n}\left(\sum_{i=1}^{k}(1+2\mathcal{B}_i)x_i \right)^{2n} = \frac{\widetilde{u}_{2n}}{(2n+1)4^n},$$

where the last equality is due to Lemma 3.4. $\qquad\square$

We have now assembled enough results to help us prove the next instrumental lemma.

Lemma 3.6. *For each integer $n \geq 0$, we have the representations*

$$T_n = \frac{1}{(n+1)2^n}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2j+1}p_1^{n-2j}u_{2j} \quad \text{and} \quad f_n = \frac{1}{(n+1)2^n}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n+1}{2j+1}p_1^{n-2j}\widetilde{u}_{2j}.$$

Proof. We proceed by induction on n. The base cases $n = 0$ and $n = 1$ are routine and hence omitted. If we revert to the umbral mechanics, in the variable $\widetilde{u}$ only, the assertion amounts to $f_n = \frac{1}{(n+1)2^n}\left\{\frac{(p_1+\widetilde{u})^{n+1}-(p_1-\widetilde{u})^{n+1}}{2\widetilde{u}}\right\}$.

The case n odd: We know from (3.6) and Lemma 3.5 that

$$f_{2n+1} = \sum_{j=1}^{2n+1}(-1)^{j+1}\binom{2n+1}{j}f_1^j f_{2n+1-j}.$$

In light of this, we may continue using the induction hypothesis and the seed $f_1 = \frac{1}{2}p_1$ to obtain

$$f_{2n+1} = \sum_{j=1}^{2n+1}\frac{(-1)^{j+1}\binom{2n+1}{j}p_1^j}{(2n+2-j)2^{2n+1}}\left\{\frac{(p_1+\widetilde{u})^{2n+2-j}-(p_1-\widetilde{u})^{2n+2-j}}{2\widetilde{u}}\right\}$$

$$= \frac{1}{(2n+2)2^{2n+1}}\sum_{j=1}^{2n+1}(-1)^{j+1}\binom{2n+2}{j}p_1^j\left\{\frac{(p_1+\widetilde{u})^{2n+2-j}-(p_1-\widetilde{u})^{2n+2-j}}{2\widetilde{u}}\right\}$$

$$= \frac{1}{(2n+2)2^{2n+1}}\left\{\frac{(p_1+\widetilde{u})^{2n+2}-(p_1-\widetilde{u})^{2n+2}}{2\widetilde{u}}\right\}.$$

The case n even: The argument is similar. We apply Lemma 3.5 and induction to write

$$f_{2n} = \frac{\widetilde{u}_{2n}}{(2n+1)4^n} + \sum_{j=1}^{2n}\frac{(-1)^{j+1}}{(2n+1)2^{2n}}\binom{2n+1}{j}p_1^j\left\{\frac{(p_1+\widetilde{u})^{2n+1-j}-(p_1-\widetilde{u})^{2n+1-j}}{2\widetilde{u}}\right\}$$

$$= \frac{1}{(2n+1)2^{2n}}\left\{\frac{(p_1+\widetilde{u})^{2n+1}-(p_1-\widetilde{u})^{2n+1}}{2\widetilde{u}}\right\}.$$

This completes the proof. $\qquad\square$

Proof of Theorem 1.3. The proof follows by combining two structural results. The first one concerning $T_n(\boldsymbol{x}^k)$ comes from Lemma 3.6. The other is our principal result from Theorem 1.2 in regard to $Y_n(q)$. Their common formulation is based on (1.10), which is due to the mapping Ψ. $\qquad\square$

3.4. **Proof of Theorem 1.7.** In this subsection, we begin by recalling some calculations found in [12, p. 59, (13)], namely that

$$\frac{T_3(\boldsymbol{x}^k)}{T_1^3(\boldsymbol{x}^k)} = 3\frac{T_2(\boldsymbol{x}^k)}{T_1^2(\boldsymbol{x}^k)} - 2,$$

$$\frac{T_5(\boldsymbol{x}^k)}{T_1^5(\boldsymbol{x}^k)} = 5\frac{T_4(\boldsymbol{x}^k)}{T_1^4(\boldsymbol{x}^k)} - 20\frac{T_2(\boldsymbol{x}^k)}{T_1^2(\boldsymbol{x}^k)} + 16,$$

$$\frac{T_7(\boldsymbol{x}^k)}{T_1^7(\boldsymbol{x}^k)} = 7\frac{T_6(\boldsymbol{x}^k)}{T_1^6(\boldsymbol{x}^k)} - 70\frac{T_4(\boldsymbol{x}^k)}{T_1^4(\boldsymbol{x}^k)} + 336\frac{T_2(\boldsymbol{x}^k)}{T_1^2(\boldsymbol{x}^k)} - 272.$$

The author [12] also derived a similar formulation for the polynomials $f_n(\boldsymbol{x}^k)$. However, a compact formula is missing from [12, (25)-(26)] and we fill that gap. For integers $n \geq 1$, the modified version of [12, page 61, (25)] is

$$(3.8) \qquad \frac{f_{2n+1}(\boldsymbol{x}^k)}{f_1^{2n+1}(\boldsymbol{x}^k)} = \sum_{j=0}^{n}(-1)^j A_{2j+1}\binom{2n+1}{2j+1}\frac{f_{2n-2j}(\boldsymbol{x}^k)}{f_1^{2n-2j}(\boldsymbol{x}^k)}.$$

Theorem 1.7 is then an immediate consequence of Theorem 1.5 and (3.8) above.

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