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The integrality of reverse Legendre polynomials

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Abstract. In the study of reverse Legendre polynomials, certain coefficients are conjectured to be integers. We apply generating functions to confirm this claim.

1. Introduction

The classical Legendre polynomials $P_n(x)$ appear from the Gram-Schmidt orthogonalization process applied to the ordered basis $\mathcal{B} = \{1, x, x^2, \ldots\}$ with the usual inner product ($\langle f, g\rangle$ is the integral over $[-1,1]$) subject to the normalization $P_n(1) = 1$. In recent work S. Weintraub [**2**] constructs a family of polynomials applying the same orthogonalization procedure and normalization, but now starting with the ordered basis $\overline{\mathcal{B}} = \{x^n, x^{n-1}, \ldots, x, 1\}$. These are the reverse Legendre polynomials. Several properties of this family are established in [**2**]. The author conjectures that the expression

$$\alpha(m,k,j) = 2\binom{m}{j} \times \beta(m,k,j) \tag{1.1}$$

where

$$\beta(m,k,j) = \frac{\binom{2m+2k+2j+1}{2m}\binom{2m}{m}}{\binom{m+k+j}{m}} \tag{1.2}$$

is always an integer. This was presented as a problem in the special session on Orthogonal Polynomials and their Applications.

The goal of this short note is to prove a slightly stronger result:

Theorem 1.1. *The expression $\beta(m,k,j)$ is an integer.*

2. The proof

Make a notational simplicity by writing $\ell = k + j$ to obtain

$$\gamma(m,\ell) := \beta(m,k,j) = \frac{\binom{2m+2\ell+1}{2m}\binom{2m}{m}}{\binom{m+\ell}{m}}. \tag{2.1}$$

The power series coming from the identity

$$(1-4x)^{-(\ell+1)} \times (1-4x)^{-1/2} = (1-4x)^{-(\ell+3/2)} \tag{2.2}$$

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yields

$$\left[\sum_{a=0}^{\infty} 4^a \binom{a+\ell}{a} x^a\right] \times \left[\sum_{b=0}^{\infty} \binom{2b}{b} x^b\right] = \sum_{m=0}^{\infty} 4^m \binom{m+\ell+\frac{1}{2}}{m} x^m. \tag{2.3}$$

Cauchy's formula for the product of the two series on the left and matching corresponding powers of x give the identity

$$\sum_{i=0}^{m} 4^i \binom{i+\ell}{i}\binom{2m-2i}{m-i} = 4^m \binom{m+\ell+\frac{1}{2}}{m}. \tag{2.4}$$

The integrality of $\gamma(m,\ell)$ follows once we show

$$\gamma(m,\ell) = 4^m \binom{m+\ell+\frac{1}{2}}{m}. \tag{2.5}$$

Since the left-hand side of (2.4) is clearly an integer, Theorem 1.1 is established.

The proof of (2.5) follows directly by elementary manipulation or by writing factorials in terms of the gamma function $\Gamma(x)$ and using the duplication formula [**1**, 8.335.1]

$$\Gamma(x+\tfrac{1}{2}) = \frac{\Gamma(2x)\sqrt{\pi}}{2^{2x-1}\Gamma(x)} \tag{2.6}$$

to write

$$\binom{r+\frac{1}{2}}{m} = \frac{(2r+1)!\,(r-m)!}{2^{2m} r! m! (2r-2m+1)!}. \tag{2.7}$$

Then (2.5) is immediate.

References

[1] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products*, 8th ed., Elsevier/Academic Press, Amsterdam, 2015. Translated from the Russian; Translation edited and with a preface by Daniel Zwillinger and Victor Moll; Revised from the seventh edition [MR2360010]. MR3307944 ↑194

[2] Steven H. Weintraub, *Reverse Legendre polynomials*, Arch. Math. (Basel) **118** (2022), no. 6, 593–604, DOI 10.1007/s00013-022-01740-2. MR4423453 ↑193

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