# ARITHMETIC PROPERTIES FOR GENERALIZED CUBIC PARTITIONS AND OVERPARTITIONS MODULO A PRIME

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Abstract. A cubic partition is an integer partition wherein the even parts can appear in two colors. In this paper, we introduce the notion of generalized cubic partitions and prove a number of new congruences akin to the classical Ramanujan-type. We emphasize two methods of proofs, one elementary (relying significantly on functional equations) and the other based on modular forms. We close by proving analogous results for generalized overcubic partitions.

## 1. Introduction

A partition $\lambda$ of a positive integer $n$ is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The values $\lambda_1, \lambda_2, \ldots, \lambda_r$ are called the parts of $\lambda$. We let $p(n)$ denote the number of partitions of $n$ for $n \geq 1$, and we define $p(0) := 1$. As an example, note that the partitions of $n = 4$ are

$$4, \;\; 3+1, \;\; 2+2, \;\; 2+1+1, \;\; 1+1+1+1$$

and this means $p(4) = 5$.

Throughout this work, we adopt the notations $(a;q)_\infty = \prod_{j\geq 0}(1 - aq^j)$, where $|q| < 1$, and $f_k := (q^k;q^k)_\infty$. As was proven by Euler, we know that

$$P(q) := \sum_{n\geq 0} p(n)\, q^n = \prod_{j\geq 1} \frac{1}{1-q^j} = \frac{1}{f_1}.$$

Because the focus of this paper is on divisibility properties of certain partition functions, we remind the reader of the celebrated Ramanujan congruences for $p(n)$ [10, p. 210, p. 230]: For all $n \geq 0$,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \quad \text{and} \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned} \tag{1.1}$$

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Indeed, among Ramanujan's discoveries, the equality

$$\sum_{n\geq 0} p(5n+4)\,q^n = 5\,\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6} \tag{1.2}$$

was regarded as his "Most Beautiful Identity" by both Hardy and MacMahon (see [10, p. xxxv]).

Based on an identity on Ramanujan's cubic continued fractions [1], Chan [3] introduced the notion of cubic partitions. A *cubic partition* of weight $n$ is a partition of $n$ wherein the even parts can appear in two colors. We denote the number of such cubic partitions by $a_2(n)$ and define $a_2(0) := 1$. For example, $a_2(3) = 4$ since the list of cubic partitions of $n = 3$ is: $3,\ 2+1,\ 2+1,\ 1+1+1$. We also compute $a_2(4) = 9$ because the cubic partitions of 4 are

$$4,\ \ 4,\ \ 3+1,\ \ 2+2,\ \ 2+2,\ \ 2+2,\ \ 2+1+1,\ \ 2+1+1,\ \ 1+1+1+1.$$

It is clear from the definition of cubic partitions that the generating function for $a_2(n)$ is given by

$$F_2(q) := \sum_{n\geq 0} a_2(n)\,q^n = \prod_{j\geq 1}\frac{1}{(1-q^j)(1-q^{2j})} = \frac{1}{f_1 f_2}.$$

Chan [3] succeeded in proving the following elegant analogue of (1.2):

$$\sum_{n\geq 0} a_2(3n+2)\,q^n = 3\,\frac{(q^3;q^3)_\infty^3 (q^6;q^6)_\infty^3}{(q;q)_\infty^4 (q^2;q^2)_\infty^4}.$$

It is immediate that, for all $n \geq 0$,

$$a_2(3n+2) \equiv 0 \pmod{3}, \tag{1.3}$$

a result similar in nature to (1.1). Since then, many authors have studied similar congruences for $a_2(n)$ (see [5], [6] and references therein). In particular, we highlight the following result of Chan and Toh [4] which was proven using modular forms:

**Theorem 1.1.** *For all $n \geq 0$,*

$$a_2(5^j n + d_j) \equiv 0 \pmod{5^{\lfloor j/2 \rfloor}}$$

*where $d_j$ is the inverse of 8 modulo $5^j$.*

By way of generalization, we define a *generalized cubic partition* of weight $n$ to be a partition of $n \geq 1$ wherein each even part may appear in $c \geq 1$ different colors. We denote the number of such generalized cubic partitions by $a_c(n)$ and define $a_c(0) := 1$. Notice that, for all $n \geq 0$,

$a_1(n) = p(n)$ while $a_2(n)$ enumerates the cubic partitions of $n$ as described above. It is clear that the generating function for $a_c(n)$ is given by

$$F_c(q) := \sum_{n\geq 0} a_c(n)\, q^n = \prod_{j\geq 1} \frac{1}{(1-q^j)(1-q^{2j})^{c-1}} = \frac{1}{f_1 f_2^{c-1}}. \tag{1.4}$$

Our primary goal in this work is to prove several congruences modulo primes satisfied by $a_c(n)$ for an infinite family of values of $c$. The proof techniques that we will utilize will include elementary approaches as well as modular forms. In particular, we will prove the following infinite family of congruences modulo a prime $p \geq 3$ which are reminiscent of (1.3):

**Theorem 1.2.** *Let $p$ be an odd prime. Then, for all $n \geq 0$,*

$$a_{p-1}(pn + r) \equiv 0 \pmod{p}$$

*where $r$ is an integer, $1 \leq r \leq p-1$, such that $8r+1$ is a quadratic nonresidue modulo $p$.*

We will then show that Theorem 1.2 can be generalized in the following natural way:

**Corollary 1.3.** *Let $p \geq 3$ be prime and let $r$, $1 \leq r \leq p-1$ such that, for all $n \geq 0$, $a_{p-1}(pn + r) \equiv 0 \pmod{p}$. Then, for any $k \geq 1$, and for all $n \geq 0$, $a_{kp-1}(pn + r) \equiv 0 \pmod{p}$.*

We will also prove two "isolated" congruences, modulo the primes 7 and 11, respectively.

**Theorem 1.4.** *For all $n \geq 0$,*

$$a_3(7n + 4) \equiv 0 \pmod{7} \quad \textit{and}$$
$$a_5(11n + 10) \equiv 0 \pmod{11}.$$

These two results will be proved via modular forms (using arguments that are very similar to one another).

The remainder of this paper is organized as follows. In Section 2 we present elementary proofs of Theorem 1.2 and Corollary 1.3. Section 3 contains the basic results from the theory of modular forms which we need to prove Theorem 1.4. We close that section with the proof of Theorem 1.4. Finally, in Section 4, we consider *generalized cubic overpartitions* and prove an infinite family of congruences for the related partition functions via elementary methods. We conclude Section 4 with some relevant remarks.

## 2. Elementary Proofs of Theorem 1.2 and Corollary 1.3

Sellers [11, Theorem 2.1] developed, among several other results, a functional equation for $F_2(q)$. That result can be generalized to our family of generating functions. Namely, thanks to (1.4), we can easily show that

$$F_c(q) = \psi(q)\psi(q^2)^{c-1}F_c(q^2)^2 \tag{2.1}$$

where

$$\psi(q) := \frac{f_2^2}{f_1} = \sum_{k\geq 0} q^{k(k+1)/2}$$

is one of Ramanujan's well–known theta functions [2, p. 6]. We can then iterate (2.1) to prove the following:

**Lemma 2.1.** *Let $p \geq 3$ be prime. Then*

$$F_{p-1}(q) = \psi(q)\prod_{i\geq 1}\psi(q^{2^i})^{p\cdot 2^{i-1}}.$$

*Proof.* We have

$$\begin{aligned}
F_{p-1}(q) &= \frac{1}{f_1 f_2^{p-2}} = \psi(q)\psi(q^2)^{p-2}F_{p-1}(q^2)^2 \\
&= \psi(q)\psi(q^2)^{p-2}\left(\psi(q^2)\psi(q^4)^{p-2}F_{p-1}(q^4)^2\right)^2 \\
&= \psi(q)\psi(q^2)^{p}\psi(q^4)^{2(p-2)}F_{p-1}(q^4)^4 \\
&= \psi(q)\psi(q^2)^{p}\psi(q^4)^{2(p-2)}\left(\psi(q^4)\psi(q^8)^{p-2}F_{p-1}(q^8)^2\right)^4 \\
&= \psi(q)\psi(q^2)^{p}\psi(q^4)^{2p}\psi(q^8)^{4(p-2)}F_{p-1}(q^8)^8 \\
&\quad\vdots
\end{aligned}$$

The result follows by continuing to iterate (2.1) indefinitely. □

Lemma 2.1 then leads to a straightforward proof of Theorem 1.2.

*Proof of Theorem 1.2.* Thanks to Lemma 2.1, we know

$$F_{p-1}(q) = \psi(q)\prod_{i\geq 1}\psi(q^{2^i})^{p\cdot 2^{i-1}} \equiv \psi(q)\prod_{i\geq 1}\psi(q^{p\cdot 2^i})^{2^{i-1}} \pmod{p}.$$

Thus, if we wish to focus our attention on $a_{p-1}(pn+r)$ with the conditions as stated in the theorem, then we only need to consider $\psi(q)$ because

$$\prod_{i\geq 1}\psi(q^{p\cdot 2^i})^{2^{i-1}}$$

is a function of $q^p$. Thus, we simply need to confirm that we have no solutions to the equation

$$pn + r = \frac{k(k+1)}{2}$$

or

$$8(pn+r)+1 = 8\left(\frac{k(k+1)}{2}\right)+1 = (2k+1)^2.$$

If we did have a solution to the above, then

$$8r + 1 \equiv (2k+1)^2 \pmod{p},$$

but we have explicitly assumed that $r$ has been selected so that $8r + 1$ is a quadratic nonresidue modulo $p$. Therefore, there can be no such solutions, and this proves our result. □

*Proof of Corollary 1.3.* The generating function for $a_{kp-1}(n)$ is

$$\frac{1}{f_1 f_2^{kp-2}} = \frac{1}{f_1 f_2^{p-2} f_2^{(k-1)p}} \equiv \frac{1}{f_1 f_2^{p-2} f_{2p}^{k-1}} \pmod{p}$$

If we wish to focus our attention on $a_{kp-1}(pn + r)$ with the conditions as stated in the corollary, then we only need to look at $\frac{1}{f_1 f_2^{p-2}}$ because $f_{2p}^{k-1}$ is a function of $q^p$. Notice that $\frac{1}{f_1 f_2^{p-2}}$ is the generating function for $a_{p-1}(n)$. Given that a congruence mod $p$ is assumed to hold for $a_{p-1}(n)$, it must be the case that a similar congruence modulo $p$ will hold for $a_{kp-1}(n)$ on the same arithmetic progression. □

## 3. A Modular Forms Proof of Theorem 1.4

We begin this section with some definitions and basic facts on modular forms that are instrumental in furnishing our proof of Theorem 1.4. For additional details, see for example [8, 9]. We first identify the matrix groups

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

$$\Gamma_0(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N) : a \equiv d \equiv 1 \pmod{N} \right\},$$

and

$$\Gamma(N) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N}, \text{ and } b \equiv c \equiv 0 \pmod{N} \right\},$$

where $N$ is a positive integer. A subgroup $\Gamma$ of the group $\mathrm{SL}_2(\mathbb{Z})$ is called a *congruence subgroup* if $\Gamma(N) \subseteq \Gamma$ for some $N$. The smallest $N$ such that $\Gamma(N) \subseteq \Gamma$ is called the *level of* $\Gamma$. For example, $\Gamma_0(N)$ and $\Gamma_1(N)$ are congruence subgroups of level $N$.

Let $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half of the complex plane. Then, the following subgroup of the general linear group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on $\mathbb{H}$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \dfrac{az + b}{cz + d}$. We identify $\infty$ with $\dfrac{1}{0}$ and define

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{r}{s} = \frac{ar + bs}{cr + ds},$$

where $\dfrac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. This gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the extended upper half-plane $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that $\Gamma$ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A *cusp* of $\Gamma$ is an equivalence class in $\mathbb{P}^1 = \mathbb{Q} \cup \{\infty\}$ under the action of $\Gamma$. The group $\mathrm{GL}_2^+(\mathbb{R})$ also acts on functions $f : \mathbb{H} \to \mathbb{C}$. In particular, suppose that $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2^+(\mathbb{R})$. If $f(z)$ is a meromorphic function on $\mathbb{H}$ and $\ell$ is an integer, then define the slash operator $|_\ell$ by

$$(f|_\ell \gamma)(z) := (\det \gamma)^{\ell/2} (cz + d)^{-\ell} f(\gamma z).$$

**Definition 3.1.** Let $\Gamma$ be a congruence subgroup of level $N$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a *modular form* with integer weight $\ell$ on $\Gamma$ if the following hold:

(1) We have $f(\gamma z) = (cz + d)^\ell f(z)$ for all $z \in \mathbb{H}$ and all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$.
(2) If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_\ell \gamma)(z)$ has a Fourier expansion of the form

$$(f|_\ell \gamma)(z) = \sum_{n \geq 0} a_\gamma(n) q_N^n,$$

where $q_N := e^{2\pi i z/N}$.

For a positive integer $\ell$, the complex vector space of modular forms of weight $\ell$ with respect to a congruence subgroup $\Gamma$ is denoted by $M_\ell(\Gamma)$.

**Definition 3.2.** [9, Definition 1.15] If $\chi$ is a Dirichlet character modulo $N$, then we say that a modular form $f \in M_\ell(\Gamma_1(N))$ has *Nebentypus character* $\chi$ if

$$f\left(\frac{az + b}{cz + d}\right) = \chi(d)(cz + d)^\ell f(z)$$

for all $z \in \mathbb{H}$ and all $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. The space of such modular forms is denoted by $M_\ell(\Gamma_0(N), \chi)$.

In this paper, the relevant modular forms are those that arise from eta-quotients. The *Dedekind eta-function* $\eta(z)$ is defined by

$$\eta(z) := q^{1/24}(q;q)_\infty = q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi iz}$ and $z \in \mathbb{H}$, the upper half-plane. A function $f(z)$ is called an *eta-quotient* if it is of the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta},$$

where $N$ is a positive integer and $r_\delta$ is an integer. We now recall two valuable theorems from [9, p. 18] which will be used to prove our results.

**Theorem 3.3.** [9, Theorem 1.64] *If $f(z) = \prod_{\delta|N}\eta(\delta z)^{r_\delta}$ is an eta-quotient such that $\ell = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$,*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \quad \text{and} \quad \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

*then $f(z)$ satisfies $f(\gamma z) = \chi(d)(cz+d)^\ell f(z)$ for every $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$. Here the character $\chi$ is defined by $\chi(\bullet) := \left(\frac{(-1)^\ell s}{\bullet}\right)$, where $s := \prod_{\delta|N}\delta^{r_\delta}$.*

Suppose that $f$ is an eta-quotient satisfying the conditions of Theorem 3.3 and that the associated weight $\ell$ is a positive integer. If the function $f(z)$ is holomorphic at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_\ell(\Gamma_0(N), \chi)$. The next theorem gives a necessary condition for determining orders of an eta-quotient at cusps.

**Theorem 3.4.** [9, Theorem 1.65] *Let $c, d$ and $N$ be positive integers with $d \mid N$ and $\gcd(c,d) = 1$. If $f$ is an eta-quotient satisfying the conditions of Theorem 3.3 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d})d\delta}.$$

We now remind ourselves a result of Sturm [12] which gives a criterion to test whether two modular forms are congruent modulo a given prime.

**Theorem 3.5.** *Let $k$ be an integer and $g(z) = \sum_{n=0}^{\infty} a(n)q^n$ a modular form of weight $k$ for $\Gamma_0(N)$. For any given positive integer $m$, if $a(n) \equiv 0 \pmod{m}$ holds for all $n \leq \frac{kN}{12}\prod_{\substack{p \text{ prime} \\ p|N}}\left(1+\frac{1}{p}\right)$, then $a(n) \equiv 0 \pmod{m}$ holds for any $n \geq 0$.*

We now recall the definition of Hecke operators. Let $m$ be a positive integer and $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_\ell(\Gamma_0(N), \chi)$. Then the action of Hecke operator $T_m$ on $f(z)$ is defined by

$$f(z) \mid T_m := \sum_{n=0}^{\infty} \left( \sum_{d|\gcd(n,m)} \chi(d) d^{\ell-1} a\left(\frac{nm}{d^2}\right) \right) q^n.$$

In particular, if $m = p$ is prime, we have

$$f(z) \mid T_p := \sum_{n=0}^{\infty} \left( a(pn) + \chi(p) p^{\ell-1} a\left(\frac{n}{p}\right) \right) q^n. \tag{3.1}$$

We take by convention that $a(n/p) = 0$ whenever $p \nmid n$. The next result follows directly from (3.1).

**Proposition 3.6.** Let $p$ be a prime, $g(z) \in \mathbb{Z}[[q]], h(z) \in \mathbb{Z}[[q^p]]$, and $k > 1$. Then, we have

$$(g(z)h(z)) \mid T_p \equiv \left( g(z) | T_p \cdot h(z/p) \right) \pmod{p}.$$

With the above in place, we are ready to supply the proof of Theorem 1.4.

*Proof of Theorem 1.4.* We begin by proving the mod 7 congruence that appears in the theorem. By taking $c = 3$ in (1.4), we have

$$\sum_{n \geq 0} a_3(n)\, q^n = \prod_{k \geq 1} \frac{1}{(1-q^k)(1-q^{2k})^2}. \tag{3.2}$$

Let

$$H(z) := \frac{\eta^{76}(z)}{\eta^2(2z)}.$$

By Theorems 3.3 and 3.4, we find that $H(z)$ is a modular form of weight 37, level 8 and character $\chi_1 = (\frac{-2^{-2}}{\bullet})$. By (3.2), the Fourier expansion of our form satisfies

$$H(z) = \left( \sum_{n=0}^{\infty} a_3(n) q^{n+3} \right) \prod_{k \geq 1} (1-q^k)^{77}.$$

Using Proposition 3.6, we calculate that

$$H(z) \mid T_7 \equiv \left( \sum_{n=0}^{\infty} a_3(7n+4) q^{n+1} \right) \prod_{k \geq 1} (1-q^k)^{11} \pmod{7}.$$

Since the Hecke operator is an endomorphism on $M_{37}(\Gamma_0(8), \chi_1)$, we have that $H(z) \mid T_7 \in M_{37}(\Gamma_0(8), \chi_1)$. By Theorem 3.5, the Sturm bound for this space of forms is 37. Using *SageMath*, we verify that the Fourier coefficients of $H(z) \mid T_7$ up to the desired bound are congruent to 0 modulo 7. Hence, Theorem 3.5 confirms that $H(z) \mid T_7 \equiv 0 \pmod{7}$. This completes the proof of the mod 7 congruence.

We next consider the mod 11 congruence in the theorem whose proof is rather similar to the proof of the mod 7 congruence above. By (1.4), we have

$$\sum_{n\geq 0} a_5(n)\, q^n = \prod_{k\geq 1} \frac{1}{(1-q^k)(1-q^{2k})^4}. \tag{3.3}$$

Let

$$G(z) := \frac{\eta^{32}(z)}{\eta^4(2z)}.$$

By Theorems 3.3 and 3.4, we find that $G(z)$ is a modular form of weight 14, level 4 and character $\chi_0 = (\frac{2^{-4}}{\bullet})$. By (3.3), the Fourier expansion of our form satisfies

$$G(z) = \left(\sum_{n=0}^{\infty} a_5(n) q^{n+1}\right) \prod_{k\geq 1} (1-q^k)^{33}.$$

Using Proposition 3.6, we calculate that

$$G(z)\,|\,T_{11} \equiv \left(\sum_{n=0}^{\infty} a_5(11n+10) q^{n+1}\right) \prod_{k\geq 1} (1-q^k)^{3} \pmod{11}.$$

Since the Hecke operator is an endomorphism on $M_{14}\left(\Gamma_0(4), \chi_0\right)$, we have that $G(z)\,|\,T_{11} \in M_{14}\left(\Gamma_0(4), \chi_0\right)$. By Theorem 3.5, the Sturm bound for this space of forms is 7. Hence, Theorem 3.5 confirms that $G(z)\,|\,T_{11} \equiv 0 \pmod{11}$. This completes the proof of the mod 11 congruence. □

## 4. Closing Thoughts

We now transition to a consideration of combinatorial objects related to generalized cubic partitions, namely, generalized cubic overpartitions.

An *overpartition* of $n$ is a partition of $n$ in which the first occurrence of a part may be overlined. In 2010, Kim [7] introduced an overpartition version of cubic partitions, sometimes called *overcubic partitions*. Following the four cubic partitions of $n = 3$ from the Introduction, we find that the corresponding 12 overcubic partitions of $n = 3$ are given by the following:

$$3,\ \ \overline{3},\ \ 2+1,\ \ {\color{red}2}+1,\ \ \overline{2}+1,\ \ {\color{red}\overline{2}}+1,$$
$$2+\overline{1},\ \ {\color{red}2}+\overline{1},\ \ \overline{2}+\overline{1},\ \ {\color{red}\overline{2}}+\overline{1},\ \ 1+1+1,\ \ \overline{1}+1+1.$$

If we now generalize these to mirror our generalization of cubic partitions, then the generating function for these cubic overpartitions is given by

$$\overline{F}_c(q) := \sum_{n\geq 0} \overline{a}_c(n)\, q^n = \frac{(-q;q)_\infty (-q^2;q^2)_\infty^{c-1}}{(q;q)_\infty (q^2;q^2)_\infty^{c-1}}.$$

We now prove the following natural companion to Corollary 1.3.

**Theorem 4.1.** *Let $p \geq 3$ be prime, and let $r$, $1 \leq r \leq p-1$, such that $r$ is a quadratic nonresidue modulo $p$. Also let $k$ be a positive integer. Then, for all $n \geq 0$, $\overline{a}_{kp-1}(pn+r) \equiv 0 \pmod{p}$.*

*Proof.* After elementary manipulations, the generating function for $\overline{a}_c(n)$ can be given in the form

$$\overline{F}_c(q) = \frac{f_4^{c-1}}{f_1^2 f_2^{2c-3}}.$$

Now consider the above generating function for $c = kp-1$. Then we have

$$\begin{aligned}\sum_{n\geq 0} \overline{a}_{kp-1}(n)q^n &= \frac{f_4^{kp-2}}{f_1^2 f_2^{2kp-5}} = \left(\frac{f_4^{kp}}{f_2^{2kp}}\right)\left(\frac{f_2^5}{f_1^2 f_4^2}\right) = \left(\frac{f_4^{kp}}{f_2^{2kp}}\right)\cdot \varphi(q) \\ &\equiv \left(\frac{f_{4p}^{k}}{f_{2p}^{2k}}\right)\cdot \varphi(q) \pmod{p}\end{aligned}$$

where

$$\varphi(q) := 1 + 2\sum_{j\geq 1} q^{j^2}$$

is another of Ramanujan's well–known theta functions [2, p. 6]. Note that

$$\frac{f_{4p}^{k}}{f_{2p}^{2k}}$$

is a function of $q^p$, which means that, in order to prove this theorem, we simply need to consider whether $pn + r = j^2$ for some $r$ and $j$. Since $r$ is assumed to be a quadratic nonresidue modulo $p$, we know that there can be no solutions to the equation $pn + r = j^2$. This proves our result. $\square$

We close this paper with two sets of brief remarks.

(1) We first return to the generalized cubic partitions and make the following elementary observation. For all $c > 1$ which satisfy $c \equiv 1 \pmod{p}$ for $p = 5, 7$ or $11$, it is clear that $a_c(n)$ "inherits" the corresponding congruence modulo 5, 7, or 11 as Ramanujan's (1.1). That is to say, for all $j \geq 0$ and $n \geq 0$,

$$\begin{aligned} a_{5j+1}(5n+4) &\equiv 0 \pmod{5}, \\ a_{7j+1}(7n+5) &\equiv 0 \pmod{7}, \quad \text{and} \\ a_{11j+1}(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

(2) As we mentioned above, our emphasis in this paper has been on congruences modulo a prime $p$ satisfied by $a_c(n)$ or $\overline{a}_c(n)$ for particular values of $c$. We encourage the interested reader to consider

identifying and proving additional congruences for these families of functions modulo powers of a prime akin to Theorem 1.1.

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