

A NOTE ON FIBONACCI-TYPE POLYNOMIALS

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ABSTRACT. We opt to study the convergence of maximal real roots of certain Fibonacci-type polynomials given by $G_n = x^k G_{n-1} + G_{n-2}$. The special cases $k = 1$ and $k = 2$ are found in [4] and [7], respectively.

In the sequel, \mathbb{P} denotes the set of positive integers. The Fibonacci polynomials [2] are defined recursively by $F_0(x) = 1$, $F_1(x) = x$ and

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x), \quad n \geq 2.$$

Fact 1. *Let $n \geq 1$. Then the roots of $F_n(x)$ are given by*

$$x_k = 2i \cos\left(\frac{\pi k}{n+1}\right), \quad 1 \leq k \leq n.$$

In particular a Fibonacci polynomial has no positive real roots.

Proof. The Fibonacci polynomials are essentially Tchebycheff polynomials. This is well-known (see, for instance [2]). \square

Let $k \in \mathbb{P}$ be fixed. Several authors ([3]-[7]) have investigated the so-called *Fibonacci-type* polynomials. In this note, we focus on a particular group of polynomials recursively defined by

$$G_n^{(k)}(x) = \begin{cases} -1, & n = 0 \\ x - 1, & n = 1 \\ x^k G_{n-1}^{(k)}(x) + G_{n-2}^{(k)}(x), & n \geq 2. \end{cases}$$

When there is no confusion, we suppress the index k to write G_n for $G_n^{(k)}(x)$. We list a few basic properties relevant to our work here.

Fact 2. *For each $k \in \mathbb{P}$, there is a rational generating function for G_n ; namely,*

$$\sum_{n \geq 0} G_n^{(k)}(x) t^n = \frac{(x^k + x - 1)t - 1}{1 - x^k t - t^2}.$$

Proof. The claim follows from the definition of G_n . \square

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Fact 3. *The following relation holds*

$$G_n^{(k)}(x) = \frac{G_{n-1}^{(k)}F_{n-1}(x^k) + (-1)^{n-1}}{F_{n-2}(x^k)}.$$

Proof. We write the equivalent formulation

$$G_n^{(k)}(x) = \det \begin{pmatrix} x-1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & x^k & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & x^k & -1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & x^k & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & x^k & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & x^k \end{pmatrix},$$

then we apply Dodgson's determinantal formula [1]. \square

Fact 4. *For a fixed k , let $\{g_n^{(k)}\}_{n \in \mathbb{P}}$ be the maximal real roots of $\{G_n^{(k)}(x)\}_n$. Then $\{g_{2n}^{(k)}\}_n$ and $\{g_{2n-1}^{(k)}\}_n$ are decreasing and increasing sequences, respectively.*

Proof. First, each g_n exists since $G_n(0) = 1 < 0$ and $G_n(\infty) = \infty$. Assume $x > 0$. Invoking Fact 3 from above, twice, we find that

$$F_{2n-3}(x^k)G_{2n}^{(k)}(x) = F_{2n-1}(x^k)G_{2n-2}^{(k)}(x) + x^k, \quad F_{2n-2}(x^k)G_{2n+1}^{(k)}(x) = F_{2n}(x^k)G_{2n-1}^{(k)}(x) - x^k.$$

From these equations and $F_n(x) > 0$ (see Fact 1), it is clear that $G_{2n-2}(x) > 0$ implies $G_{2n}(x) > 0$; also if $G_{2n-2}(x) = 0$ then $G_{2n}(x) > 0$. Thus $g_{2n-2} > g_{2n}$. A similar argument shows $g_{2n+1} > g_{2n-1}$. The proof is complete. \square

Define the quantities

$$\begin{aligned} \alpha(x) &= \frac{x + \sqrt{x^2 + 4}}{2}, & \beta(x) &= \frac{x - \sqrt{x^2 + 4}}{2}, \\ p(x) &= \frac{(x-1) + \beta(x^k)}{\alpha(x^k) - \beta(x^k)}, & q(x) &= \frac{(x-1) + \alpha(x^k)}{\alpha(x^k) - \beta(x^k)}. \end{aligned}$$

Fact 5. *For $n \geq 0$ and $k \in \mathbb{P}$, we have the explicit formula*

$$G_n^{(k)}(x) = p(x)\alpha^n(x^k) - q(x)\beta^n(x^k).$$

Proof. This is the standard generalized Binet formula. \square

For each $k \in \mathbb{P}$, we introduce another set of polynomials

$$H^{(k)}(x) = x^k - x^{k-1} + x - 2.$$

Fact 6. For each $k \in \mathbb{P}$, the polynomial $H^{(k)}(x)$ has exactly one positive real root $\xi^{(k)} > 1$.

Proof. Since $H^{(k)}(x) = (x-1)(x^{k-1}+1) - 1 < 0$, whenever $0 < x \leq 1$, there are no roots in the range $0 < x \leq 1$. On the other hand, $H^{(k)}(1) < 0$, $H^{(k)}(\infty) = \infty$ and the derivative

$$\frac{d}{dx}H^{(k)}(x) = x^{k-1}(k(x-1)+1) + 1 > 0 \quad \text{whenever } x \in \mathbb{P},$$

suggest there is only one positive root (necessarily greater than 1). \square

Fact 7. If k is odd (even), then $H^{(k)}(x)$ has no (exactly one) negative real root.

Proof. For k odd, $H^{(k)}(-x) = (-x-1)(x^{k-1}+1) - 1 < 0$. For k even, $H^{(k)}(-x) = x^k + x^{k-1} - x - 2$ changes sign only once. We now apply Descartes' Rule to infer the claim. \square

Now, we state and prove the main result of the present note.

Theorem. Preserve the notations of Facts 4 and 6. Then, depending on the parity of n , the roots $\{g_n^{(k)}\}_n$ converge from above or below so that $g_n^{(k)} \rightarrow \xi^{(k)}$ as $n \rightarrow \infty$. Note also $\xi^{(k)} \rightarrow 1$ as $k \rightarrow \infty$.

Proof. For notational brevity, we suppress k and write g_n and ξ . From $G_n(g_n) = 0$ and Fact 5 above, we resolve

$$(1) \quad \frac{p(g_n)}{q(g_n)} = \frac{\beta^n(g_n^k)}{\alpha^n(g_n^k)}, \quad \text{or} \quad \frac{2(g_n - 1) + g_n^k - \sqrt{g_n^{2k} + 4}}{2(g_n - 1) + g_n^k + \sqrt{g_n^{2k} + 4}} = (-1)^n \left(1 - \frac{2g_n}{g_n^k + \sqrt{g_n^{2k} + 4}} \right)^n.$$

Using Gershgorin's Circle theorem, it is easy to see that $1 \leq g_n \leq 2$. When combined with Fact 4, the monotonic sequences $\{g_{2n}\}_n$ and $\{g_{2n-1}\}_n$ converge to finite limits, say ξ_+ and ξ_- respectively. The right-hand side of (1) vanishes in the limit $n \rightarrow \infty$, thus

$$2(\xi - 1) + \xi^k - \sqrt{\xi^{2k} + 4} = 0.$$

Further simplification leads to $H^{(k)}(\xi) = \xi^k - \xi^{k-1} + \xi - 2 = 0$. From Fact 6, such a solution is unique. So, $\xi_+ = \xi_- = \xi$ completes the proof. \square

Further Comments

In this section, we discuss the *bivariate Fibonacci* polynomials, of the *first kind* (BFP1), defined as

$$g_n(x, y) = xg_{n-1}(x, y) + yg_{n-2}(x, y), \quad g_0(x, y) = x, \quad g_1(x, y) = y.$$

If $x = y = 1$ then the resulting sequence is the Fibonacci numbers.

The following is a generating function for the BFP1

$$\sum_{n \geq 0} g_n(x, y)t^n = \frac{x + (y - x^2)t}{1 - xt - yt^2}.$$

It is also possible to give an explicit expression

$$g_n(x, y) = \sum_{k \geq 1} \frac{2n - 3k + 1}{n - k} \binom{n - k}{k - 1} x^{n-2k+1} y^k.$$

This shows clearly that each BFP1 has nonnegative coefficients.

The other variant that appears often in the literature is what we call *bivariate Fibonacci* polynomials, of the *second kind* (BFP2). These are recursively defined as

$$f_n(x, y) = x f_{n-1}(x, y) + y f_{n-2}(x, y), \quad f_0(x, y) = y, \quad f_1(x, y) = x.$$

Obviously $f_n(1, 1)$ yields the Fibonacci numbers. We also find the ordinary generating function

$$\sum_{n \geq 0} f_n(x, y) t^n = \frac{y + (x - xy)t}{1 - xt - yt^2}.$$

One interesting contrast between the two families is the following. While the roots of $f_n(x, 1)$ are all imaginary, the roots of $g_n(1, y)$ are all real numbers.

Using the corresponding generating functions for BFP2 $f_n(x, y)$ and the classical Fibonacci polynomials $F_n(x) = f_n(x, 1)$ proves the below affine relation

$$f_n(x, y^2) = xy^{n-1} F_{n-1}(x/y) + y^{n+2} F_{n-2}(x/y).$$

In particular, the *Jacobsthal-Lucas* numbers $J_n = f_n(2, 1)$ can be expressed in terms of values of the Fibonacci polynomials, at $1/\sqrt{2}$, namely that

$$J_n = 2^{\frac{n-1}{2}} F_{n-1}\left(\frac{1}{\sqrt{2}}\right) + 2^{\frac{n}{2}+1} F_{n-2}\left(\frac{1}{\sqrt{2}}\right).$$

This translates to the identity

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{5n - 9k}{n - k} \binom{n - k}{k} 2^k = 2^{n+1} + (-1)^{n+1}, \quad \text{for } n \in \mathbb{P}.$$

Since we have

$$\sum_{n \geq 0} F_n(x) t^n = \frac{1}{1 - xt - t^2} \quad \text{and} \quad F_n(x) = \sum_{k \geq 0} \binom{n - k}{k} x^{n-2k},$$

we obtain

$$f_n(x, y^2) = \sum_{k \geq 0} \binom{n - k - 1}{k} x^{n-2k} y^k + \sum_{k \geq 0} \binom{n - k - 2}{k} x^{n-2k-2} y^{k+2}.$$

In particular, when $x = 1$ there holds

$$f_n(1, y) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} \frac{(n - k - 1)!}{k!(n - 2k + 2)} Q(n, k) y^k$$

where $Q(n, k) = n^3 - 3(2k - 1)n^2 + (13k(k - 1) + 2)n - k(k - 1)(9k - 4)$.

If we alter the definition of BFP2 and specialize as $h_0 = 2, h_1 = 1, h_n(x) = h_{n-1}(x) + x h_{n-2}(x)$ then the resulting sequence of polynomials become intimately linked to the Lucas polynomials $L_n(x)$ and the so-called Jacobsthal-Lucas polynomials $J_n(x)$ as follows

$$L_n(x) = x^n h_n(1/x^2), \quad \text{and} \quad J_n(x) = h_n(2x).$$

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