On matrices that are not similar to a Toeplitz matrix and a family of polynomials

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Abstract. A conjecture from the second author's paper [Linear Algebra Appl., 332–334 (2001) 519-531] concerning a family of polynomials is proved and strengthened. A consequence of this is that for any $n > 4$ there is an $n \times n$ matrix that is not similar to a Toeplitz matrix, which was proved before for odd n and $n = 6, 8, 10$.

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1. Introduction

In the paper [4] D.S. Mackey, N. Mackey and S. Petrovic posed and studied the inverse Jordan structure problem for complex Toeplitz matrices. They showed, in particular, that every $n \times n$ complex nonderogatory matrix is similar to an upper Hessenberg Toeplitz matrix, with ones on the subdiagonal. Such a choice guarantees uniqueness of the unit upper Hessenberg Toeplitz matrix. This result was recently extended by Willmer [6], who showed that a block companion matrix is similar to a unique block unit Hessenberg matrix.

The authors [4] also investigated the problem of what happens if the nonderogatority condition is dropped and asked the question, *"Is every complex matrix similar to a Toeplitz matrix?"* This poses the inverse Jordan structure problem for Toeplitz matrices - which Jordan forms are achievable by Toeplitz matrices. Then, [4] gave an affirmative answer to this question for matrices of order $n \leq 4$ and conjectured that this might be true for all n. It is worth noting that the inverse eigenvalue question for *real* symmetric $n \times n$ Toeplitz matrices was posed in 1983 by Delsarte and Genin [1] and resolved by them for $n \leq 4$; the general case was settled only recently by Landau [3]. Landau's non-constructive proof uses topological degree theory to show that any list of n real numbers can be realized as the spectrum of an $n \times n$ real symmetric Toeplitz matrix.

In [2] the second author of the present note showed that there are matrices that are not similar to a Toeplitz matrix. Examples for such matrices are

$$\bigoplus_{j=1}^{m}(S_2 \oplus c) \qquad \text{and} \qquad \bigoplus_{j=1}^{m-2}(S_2 \oplus S_3)$$

for all $m > 1$ and $c \neq 0$. Here S_k denotes the $k \times k$ matrix of the forward shift, i.e.

$$S_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

and $\oplus$ stands for the direct sum. Note that the order of the first set of these matrices is $2m+1$ and the second matrix is nilpotent. That means that for any odd integer $n > 4$ there is an $n \times n$ matrix that is not similar to a Toeplitz matrix.

For even n the problem is more complicated. Candidates for matrices that are not similar to a Toeplitz matrix are

$$\bigoplus_{j=1}^{m-1}(S_2 \oplus 0 \oplus c) \qquad \text{and} \qquad \bigoplus_{j=1}^{m-2}(S_2 \oplus S_3 \oplus 0)\ , \tag{1.1}$$

where $c \neq 0$ and $m > 2$. It was proved in [2] that these matrices are really not similar to a Toeplitz for $m = 3, 4, 5$, that means for matrices of order 6, 8 and 10. For the general case the problem was reduced to the property of a class of polynomials defined as follows:

$$p_0(t) = p_1(t) = 1\ , \quad p_2(t) = t\ , \quad p_j(t) = -\frac{1}{2}\sum_{k=1}^{j-1} p_k(t)p_{j-k}(t) \quad (j > 2)\ . \tag{1.2}$$

It was shown that the matrices (1.1) are not similar to a Toeplitz matrix if the following is true.

Condition 1.1. *([2], p.528). For $m > 3$, the system of $m-2$ equations*

$$p_{m+2}(t) = p_{m+3}(t) = \cdots = p_{2m-1}(t) = 0$$

has only the trivial solution $t = 0$.

In the present note we show that this condition is always satisfied. Even more, the following is shown, which is the main result of the paper.

Theorem 1.2. *For $m > 1$, $p_{m+1}(t) = p_m(t) = 0$ has only the trivial solution $t = 0$.*

A consequence of this theorem is the following.

Corollary 1.3. *For any $m > 4$ there is an $m \times m$ matrix that is not similar to a Toeplitz matrix.*

2. On a family of polynomials

First we compute the generating function of the family of polynomials $\{p_j(t)\}$ defined by (1.2), which is

$$p(z,t) = \sum_{j=0}^{\infty} p_j(t) z^j .$$

Lemma 2.1. *The generating function $p(z,t)$ is given by*

$$p(z,t) = \left(1 + 2z + z^2(2t+1)\right)^{1/2} . \tag{2.1}$$

Proof. According to the definition of $p_j(t)$ we have

$$\sum_{i+k=j} p_i(t) p_k(t) = 0$$

for $j > 2$. That means that the coefficients of z^j in the expansion of $(p(z,t))^2$ in powers of z vanish if $j > 2$. Hence $p(z,t)^2$ is a quadratic polynomial in z, i.e. $p(z,t) = A(t) + B(t)z + C(t)z^2$. Taking the definition of $p_j(t)$ for $j = 0, 1, 2$ into account we obtain

$$A(t) = 1, \quad B(t) = 2, \quad C(t) = 2t + 1 ,$$

which completes the proof. □

Expanding $p(z,t)$ in powers of z we obtain the following explicit representation of $p_j(t)$[1]:

$$p_j(t) = \sum_{k=0}^{\lfloor j/2 \rfloor} 2^{j-2k} \binom{1/2}{j-k} \binom{j-k}{k} (2t+1)^k , \tag{2.2}$$

where $\lfloor j/2 \rfloor$ is the integer part of $j/2$.

The key for proving Theorem 1.2 is the following lemma.

Lemma 2.2. *The polynomials $p_j(t)$ $(j = 0, 1, \dots)$ satisfy the 3-term recursion*

$$(j+2)p_{j+2}(t) + (2j+1)p_{j+1}(t) + (j-1)(2t+1)p_j(t) = 0. \tag{2.3}$$

Proof. Let $h(z,t)$ denote the generating function of the polynomial family $\{p_j(t)\}$ defined by (2.3) with initial conditions $p_0(t) = p_1(t) = 1$. We show that $h(z,t) = p(z,t)$. Let h' denote the partial derivative of $h(z,t)$ by z and $h = h(z,t)$.

[1]A typo in [2] p.528 is corrected here. The expression is never used to affect the results of [2].

We have

$$\begin{aligned}\sum_{j=0}^{\infty}(j+2)p_{j+2}z^{j+1} &= h'-1\ ,\\ \sum_{j=0}^{\infty}(2j+1)p_{j+1}z^{j+1} &= 2zh'-h+1\ ,\\ \sum_{j=0}^{\infty}(j-1)p_{j}z^{j+1} &= z^2h'-zh\ .\end{aligned}$$

Summing up we obtain the ordinary differential equation

$$(1+2z+(2t+1)z^2)h'-(1+(2t+1)z)h=0\ .$$

As it is easily checked, the generating function $p(z,t)$ also satisfies this equation. Since $p(0,t)=h(0,t)$, we conclude that $p(z,t)=h(z,t)$. □

An alternative way to prove the lemma is to employ the explicit expression (2.2) for $p_j(t)$. This appears in the Appendix section.

Proof of Theorem 1.2. The theorem can be proved now by induction in a standard fashion. The base case, $m=2$, is evident since $p_2=t=p_3=0$ iff $t=0$. Assume the theorem is valid for $m>1$, then we claim the same is true for $m+1$. Suppose not! i.e. $p_{m+2}(\tau)=p_{m+1}(\tau)=0$ for some $\tau\neq 0$. Then Lemma 2.2 implies that $\tau=-\frac{1}{2}$. Once again, make application of the recurrence (2.3) but this time re-index m by $m-1$ to get

$$(m+1)p_{m+1}(\tau)+(2m-1)p_m(\tau)+(m-2)(2\tau+1)p_{m-2}(\tau)=0. \qquad (2.4)$$

So, $p_m(-\frac{1}{2})=0$. Hence both p_{m+1} and p_m vanish at $-\frac{1}{2}$. This contradiction to the induction step proves the theorem. □

Let us finally mention two consequences of our result. The following is immediate from Theorem 1.2 where variables are switched $w=\frac{b}{2a}z$ and the value $t=\frac{4ac}{b^2}-1$ is selected. The case $b=0$ is treated separately. It is important that $t\neq 0$.

Corollary 2.3. *Let $f(w)=(a+bw+cw^2)^{\frac{1}{2}}$, where $a\neq 0$ and $b^2-4ac\neq 0$, and $f(w)=\sum_{k=0}^{\infty}f_kz^k$ be its Maclaurin expansion. Then for all j, f_j and f_{j+1} cannot both vanish.*

The following is an equivalent formulation of Condition 1.1.

Corollary 2.4. *For $n>4$ there is no polynomial $P(t)$ of degree n such that $P(t)^2=q(t)+t^{2n-1}r(t)$ for quadratic polynomials $q(t)$ and $r(t)$, except for the trivial cases $P(t)=a+bt$ and $P(t)=at^{n-1}+bt^n$.*

Proof. Compare proof of Lemma 6.1 in [2] where the polynomials $p_j(t)$ take the place of u_k. Then, convert u_k via u_k/u_1^k. □

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3. Appendix

We show a scheme on how to arrive at the recursion

$$(j+2)p_{j+2}(t) + (2j+1)p_{j+1}(t) + (j-1)(2t+1)p_j(t) = 0 \tag{3.1}$$

for the explicit expression

$$p_j(t) = \sum_{k=0}^{\lfloor j/2 \rfloor} 2^{j-2k} \binom{1/2}{j-k} \binom{j-k}{k} (2t+1)^k$$

of the sequence $\{p_j(t)\}_j$. The idea utilizes the so-called *Wilf-Zeilberger* (WZ) method of proof [5].

Let $F(j,k) := 2^j \binom{1/2}{j-k}\binom{j-k}{k}(2t+1)^k$, and $G(j,k) := -2\frac{(j-1)(2j-2k-1)k}{(j+1-2k)(j+2-2k)}F(j,k)$.

Then one can check, preferably using a symbolic software, that

$$(j+2)F(j+2,k)+(2j+1)F(j+1,k)+(j-1)(2t+1)F(j,k) = G(j,k+1)-G(j,k).$$

Telescoping: Sum over all $-\infty < k < \infty$ and observe that

$$\sum_{k=-\infty}^{\infty} F(j,k) = \sum_{k=0}^{\lfloor j/2 \rfloor} F(j,k) = p_j(t) \quad \text{while} \quad \sum_{k=-\infty}^{\infty} G(j,k+1) = \sum_{k=-\infty}^{\infty} G(j,k),$$

since $G(j,k)$ has compact support. Then assertion (3.1) follows.

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