"TRIVIALIZING" GENERALIZATION OF SOME IZERGIN-KOREPIN-TYPE DETERMINANTS

Tewodros Amdeberhan, Doron Zeilberger

Tulane Univ., Dept of Math, New Orleans, LA 70118, USA
Rutgers Univ., Dept of Math, Piscataway, NJ 08854, USA

Abstract. We generalize (and hence trivialize and routinize) numerous explicit evaluations of determinants and Pfaffians due to Kuperberg, as well as a determinant of Tsuchiya. The level of generality of our statements render their proofs easy and routine, by using Dodgson Condensation and/or Krattenthaler's factor exhaustion method.

1. Introduction

Mills, Robbins, and Rumsey [MRR] conjectured the formula

$$A(n) = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!(n+2)!\cdots(2n-1)!} \tag{1}$$

for the total number of $n \times n$ *Alternating-sign matrices* **(ASM).** A matrix is called **ASM** if the entries are either $0, 1$ or -1, such that the non-zero entries alternate in sign in each row and column, also that the first and last non-zero entry in each row and column is 1.

The first proof of (1) was given in 1995 by Zeilberger [Z2]. Later Kuperberg [Ku2] found a different demonstration using a connection with quantum algebra and statistical mechanics: the Yang-Baxter equation. The idea emerged from Izergin and Korepin discovering a determinant formula for the partition function of *square ice* with domain wall boundary conditions, then [Ku2] observed this to be equivalent to some weighted enumerations of the **ASMs.**

There is a long list of names and acronyms for other symmetry classes of alternating-sign matrices, *VSAMS, HTSASM, QTSASM, UASM, UUASM, OSASM, VHPASM, OSASM, OOSASM, UOSASM, DSASM, DASASM, TSASM.* For instance, refer to [Ku1] for definitions and bijections with square ice (*six-vertex state*). We mention one such class which is related to our discussion here: **UASMSs.** These matrices were first considered by Tsuchiya [T] and are defined as $2n \times 2n$ **ASMs** with a right **U-turn** boundary (see [Ku1] for details).

2. Main results

Assume each matrix below lies embedded inside an infinite matrix. In this section, we have compiled a list of some new and others as generalizations of the determinant and Pfaffian evaluations of Izergin-Korepin for ASMs and the Tsuchiya determinant for **USAMs** [T].

The first theorem generalizes formulas found in Kuperberg [Ku1] (Theorem 15), as well as older determinants, mentioned there, due to Cauchy, Stembridge, Laskov- Lascoux-Thorup. Our indicated proofs are

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much more succinct and automatable, since their generality enables an easy induction using Dodgson's rule [D], or by employing Krattenthaler's elegant factor exhaustion method [Kr].

Theorem 1:

$$\det\left(\frac{1}{x_{i+a}+y_{j+b}+Ax_{i+a}y_{j+b}}\right)_{i,j}^{1,n}=\frac{\prod_{i<j}(x_{j+a}-x_{i+a})(y_{j+b}-y_{i+b})}{\prod_{i,j}^{n}(x_{i+a}+y_{j+b}+Ax_{i+a}y_{j+b})}.$$

$$\det\left(\frac{1}{x_{i+a}+y_{j+b}}-\frac{1}{1+x_{i+a}y_{j+b}}\right)_{i,j}^{1,n}$$
$$=\frac{\prod_{i+a<j+b}(1-x_{i+a}x_{j+a})(1-y_{i+b}y_{j+b})(x_{j+a}-x_{i+a})(y_{j+b}-y_{i+b})}{\prod_{i,j}^{n}(x_{i+a}+y_{j+b})(1+x_{i+a}y_{j+b})}\cdot\prod_{i=1}^{n}(1-x_{i+a})(1-y_{i+b}).$$

$$\det\left(\frac{Ay_{j+b}+Bx_{i+a}}{y_{j+b}+x_{i+a}}\right)_{i,j}^{1,n}$$
$$=(A-B)^{n-1}\frac{\left(A\prod_{j}^{n}y_{j+b}+(-1)^{n-1}B\prod_{i}^{n}x_{i+a}\right)\prod_{i+a<j+b}(x_{i+a}-x_{j+a})(y_{i+b}-y_{j+b})}{\prod_{i,j}^{n}(x_{i+a}+y_{j+b})}.$$

$$\det\left(\frac{y_{j+b}-x_{i+a}}{1-x_{i+a}y_{j+b}}\right)_{i,j}^{1,n}$$
$$=\frac{\left(\prod_{j}^{n}(1-x_{j+a})(1+y_{j+b})+(-1)^{n}\prod_{i}^{n}(1+x_{i+a})(1-y_{i+b})\right)\prod_{i+a<j+b}(x_{i+a}-x_{j+a})(y_{i+b}-y_{j+b})}{2\prod_{i,j}^{n}(1-x_{i+a}y_{j+b})}.$$

$$\det\left(\frac{1-x_{i+a}y_{j+b}}{y_{j+b}-x_{i+a}}\right)_{i,j}^{1,n}$$
$$=(-1)^{\binom{n}{2}}\frac{\left(\prod_{j}^{n}(1+x_{j+a})(1-y_{j+b})+\prod_{i}^{n}(1-x_{i+a})(1+y_{i+b})\right)\prod_{i<j}(x_{i+a}-x_{j+a})(y_{i+b}-y_{j+b})}{2\prod_{i,j}^{n}(y_{j+b}-x_{i+a})}.$$

Sketch of Proof: Automatic application of Dodgson Condensation [D, AZ]. □

Corollary (Cauchy, Stembridge, Laksov-Lascoux-Thorup)

$$\det\left(\frac{1}{x_i+y_j}\right)_{i,j}^{1,n}=\frac{\prod_{i<j}(x_j-x_i)(y_j-y_i)}{\prod_{i,j}^{n}(x_i+y_j)}.$$

$$\det\left(\frac{1}{x_i+y_j}-\frac{1}{1+x_iy_j}\right)_{i,j}^{1,n}=\frac{\prod_{i<j}(1-x_ix_j)(1-y_iy_j)(x_j-x_i)(y_j-y_i)}{\prod_{i,j}^{n}(x_i+y_j)(1+x_iy_j)}\cdot\prod_{i=1}^{n}(1-x_i)(1-y_i).$$

$$Pf^2=\det\left(\frac{x_j-x_i}{x_j+x_i}\right)_{i,j}^{1,2n}=\prod_{i<j\le 2n}\frac{(x_i-x_j)^2}{(x_i+x_j)^2}.$$

$$Pf^2=\det\left(\frac{x_j-x_i}{1-x_ix_j}\right)_{i,j}^{1,2n}=\prod_{i<j\le 2n}\frac{(x_i-x_j)^2}{(1-x_ix_j)^2}.$$

Remark: Notice that the latter two statements apply only to even-dimensional matrices. An error from [Ku1] in the second formula has been corrected here. Pf stands for *Pfaffian* of a matrix.

The next theorem generalizes, and presents variations of, several of the determinants that appear in Theorems 16 and 17 [Ku1] (typos corrected in [Kr2, Theorems 13 and 14]). Below, Z_1, Z_2, Z_3, Z_4, Z_5 are defined by (here $\gamma(a,b) = a-b, \tau(a,b) = a+b$)

$$Z_1(p,q;x,y)_{i,j} = \frac{\gamma(q^{j-i},x^{j-i})}{\tau(p^{j-i},y^{j-i})}, \qquad Z_2(p,q;x,y)_{i,j} = \frac{\gamma(q^{1+j-i},x^{1+j-i})}{\tau(p^{1+j-i},y^{1+j-i})},$$

$$Z_3(p,q;x,y)_{i,j} = \frac{\gamma(q^{-1+j-i},x^{-1+j-i})}{\tau(p^{-1+j-i},y^{-1+j-i})}, \qquad Z_4(q,q;x,x)_{i,j} = \frac{\gamma(q^{a+j-i},x^{a+j-i})}{\tau(q^{a+j-i},x^{a+j-i})},$$

$$Z_5(q,q;x,x)_{i,j} = \frac{\tau(q^{b+j-i},x^{b+j-i})}{\gamma(q^{b+j-i},x^{b+j-i})},$$

for $a \in \mathbb{Z}, b = \pm n, \pm(n+1), \ldots$.

Let $\delta_{e,n} = \frac{1+(-1)^n}{2}$ denote *Kronecker's delta* function centered at the even integers, and let $\lambda_{i,j} = 1$, if $i \neq j$ and $\lambda_{i,i} = 0$.

Theorem 2: *Write* $\gamma(a,b) = a-b, \tau(a,b) = a+b$, *we have the matrix determinants*

$$\det\left(\frac{\gamma(q^{n+j-i},x^{n+j-i})}{\gamma(p^{n+j-i},y^{n+j-i})}\right)_{i,j}^{1,n} = (py)^{\binom{n}{3}}\gamma(q,x)^n \frac{\prod_{j>i}\gamma(p^{j-i},y^{j-i})^2\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{\prod_{i,j}\gamma(p^{n+j-i},y^{n+j-i})}.$$

$$\det\left(\frac{\tau(q^{j-i},x^{j-i})}{\tau(p^{j-i},y^{j-i})}\right)_{i,j}^{1,n} = \frac{\prod_{2|j-i>0}\gamma(p^{j-i},y^{j-i})^2\prod_{2\nmid j-i>0}\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{(qx)^{\lfloor n^2/4\rfloor}\prod_{j>i}\tau(p^{j-i},y^{j-i})^2}.$$

$$\det(Z_1)_{i,j}^{1,n} = \delta_{e,n}\frac{\gamma(q,x)^n(py)^{n/2}}{(qx)^{\lfloor n^2/4\rfloor}}\frac{\prod_{2|j-i>0}\gamma(p^{j-i},y^{j-i})^2\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{\prod_{j>i}\tau(p^{j-i},y^{j-i})^2}.$$

$$\det(Z_2)_{i,j}^{1,n} = \frac{\gamma(q,x)^n}{(qx)^{\lfloor (n-1)^2/4\rfloor}}\frac{\prod_{2|j-i>0}\gamma(p^{j-i},y^{j-i})^2\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{\tau(p^n,y^n)^{1-\delta_{e,n}}\prod_{j>i}\tau(p^{j-i},y^{j-i})^2}.$$

$$\det(Z_3)_{i,j}^{1,n} = \frac{(-py)^n\gamma(q,x)^n}{(qx)^{\lfloor (n+1)^2/4\rfloor}}\frac{\prod_{2|j-i>0}\gamma(p^{j-i},y^{j-i})^2\gamma(qp^{j-i},xy^{j-i})\gamma(xp^{j-i},qy^{j-i})}{\tau(p^n,y^n)^{1-\delta_{e,n}}\prod_{j>i}\tau(p^{j-i},y^{j-i})^2}.$$

$$\det(Z_4)_{i,j}^{1,n} = 2^{n-1}\frac{q^{na}+(-1)^n x^{na}}{(qx)^{n(n-1)(n+1-3a)/6}}\frac{\prod_{j\neq i}\gamma(q^{|j-i|},x^{|j-i|})}{\prod_{i,j}\tau(q^{a+j-i},x^{a+j-i})}.$$

$$\det(Z_5)_{i,j}^{1,n} = (-1)^{\binom{n}{2}}2^{n-1}\frac{q^{nb}+x^{nb}}{(qx)^{n(n-1)(n+1-3b)/6}}\frac{\prod_{j\neq i}\gamma(q^{|j-i|},x^{|j-i|})}{\prod_{i,j}\gamma(q^{b+j-i},x^{b+j-i})}.$$

$$Pf^2 = \det\left(\lambda_{i,j}\frac{\gamma(q^{j-i},x^{j-i})\gamma(r^{j-i},z^{j-i})}{\gamma(p^{j-i},y^{j-i})}\right)_{i,j}^{1,2n} = \frac{(yp)^n}{(qxrz)^{n^2}}\frac{\prod_{j\neq i}^{1,n}\gamma(p^{|j-i|},y^{|j-i|})^2}{\prod_{i,j}^{1,n}\gamma(p^{n+j-i},y^{n+j-i})^2}\times$$
$$\prod_{i,j}^{1,n}\gamma(qp^{|j-i|},xy^{|j-i|})\gamma(xp^{|j-i|},qy^{|j-i|})\gamma(rp^{|j-i|},zy^{|j-i|})\gamma(zp^{|j-i|},ry^{|j-i|}).$$

Sketch of Proof: Identities Z_4 and Z_5 are directly amenable to Dodgson's Condensation technique [AZ]. For the remaining assertions, use the factor exhaustion method [Kr1] (see also [Ku1]): the essential idea is to compare zeros and poles on both sides of the equation at hand. We leave the straightforward details to the reader. □

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