INFINITE LOGCONVEXITY

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Abstract. A criteria to verify logconvexity of sequences is presented. Iterating this criteria produces infinitely logconvex sequences. As an application, several classical examples of sequences arising in Combinatorics and Special Functions. The paper concludes with a conjecture regarding coefficients of chromatic polynomials.

1. Introduction

Questions about the ordering of a sequence of non-negative real numbers $\mathtt{a} = \{a_k\}_k$, for $0 \leq k \leq n$, have appeared in the literature since Newton. He established that if $P(x)$ is a polynomial, all of whose zeros are real and negative, then the sequence of its coefficients $\mathtt{a} = \{a_k\}_k$ is `logconcave`; that is, $a_k^2 - a_{k-1}a_{k+1} \geq 0$ for $1 \leq k \leq n-1$. A weaker condition on sequences is that of `unimodality`: that is, there is an index r such that $a_0 \leq a_1 \leq \cdots \leq a_r \geq a_{r+1} \geq \cdots \geq a_n$. An elementary argument shows that a logconcave sequence must be unimodal. A sequence $\mathtt{a} = \{a_k\}_k$ is called `logconvex` if $a_k^2 - a_{k-1}a_{k+1} \leq 0$ for $1 \leq k \leq n-1$.

These concepts can be expressed in terms of the operator $\mathtt{a} \mapsto \mathcal{L}(\mathtt{a})$ defined by $\mathcal{L}(\mathtt{a})_k = a_k^2 - a_{k-1}a_{k+1}$. In this notation, the sequence $\mathtt{a} = \{a_k\}_k$ is `logconcave` if it satisfies $\mathcal{L}(\mathtt{a})_k \geq 0$. Similarly, the sequence is `logconvex` if $\mathcal{L}(\mathtt{a})_k \leq 0$. Iteration of $\mathcal{L}$ leads to the notion of ℓ`-logconcave` sequences, defined by the property that the sequences $\mathcal{L}^j(\mathtt{a})$ are all nonpositve for $1 \leq j \leq \ell$ and $\mathtt{a}$ is `infinitely logconvex` if it is ℓ-logconvex for every $\ell \in \mathbb{N}$. The definitions of ℓ`-logconcave` and `infinitely logconcave` are similar.

The results presented here originate with the sequence of coefficients $\{d_i(n)\}_i$ of the polynomial

$$P_n(a) = \sum_{i=0}^{n} d_i(n)a^i, \tag{1.1}$$

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defined by

$$d_i(n) = 2^{-2n} \sum_{k=i}^{n} 2^k \binom{2n-2k}{n-k} \binom{n+k}{n} \binom{k}{i}. \tag{1.2}$$

This polynomial appears in the evaluation of a definite integral. More details are presented in Section 5.

The goal of the present work is to develop a criteria which verifies the logconvexity of a variety of classical sequences. We record an elementary observation of independent interest.

Lemma 1.1. *A positive sequence* $\mathtt{a} = \{a_k\}_k$ *is logconvex if and only if* $\mathtt{a}^{-1} = \{1/a_k\}_k$ *is logconcave.*

Proof. Simply observe that

$$\mathcal{L}\left(\frac{1}{a_k}\right) = \frac{1}{a_{k-1}a_{k+1}} - \frac{1}{a_k^2} = \frac{\mathcal{L}(\mathtt{a})_k}{a_{k-1}a_k^2 a_{k+1}}. \tag{1.3}$$

□

`Remark`. This does not extend to k-logconcavity for $k \geq 2$. For instance, the sequence $\{1, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{31}\}$ is 2-logconvex but the sequence of reciprocals is not 2-logconcave.

2. The criteria

In this section we establish the basic criteria used to establish infinite logconvexity of sequences.

Proposition 2.1. *Let* $\mathtt{a} = \{a_k\}_k$*, with* $a_k = \int_X f^k(x)\, d\mu(x)$ *for a certain positive function* f *on a measure space* (X, μ)*. Then* $\mathtt{a} = \{a_k\}_k$ *is infinitely logconvex.*

Proof. It suffices to prove that $\mathcal{L}(\mathtt{a})_k \geq 0$. The general statement follows by iteration of the argument. The initial step is a consequence of

$$\begin{aligned}
\mathcal{L}(\mathtt{a})_k &= a_{k-1}a_{k+1} - a_k^2 \\
&= \int_{X\times X} f^{k-1}(x)f^{k+1}(y)d\mu(x)d\mu(y) - \int_{X\times X} f^k(x)f^k(y)d\mu(x)d\mu(y) \\
&= \frac{1}{2}\int_{X\times X} f^k(x)f^k(y)\left(\frac{f(x)}{f(y)} + \frac{f(y)}{f(x)} - 2\right) d\mu(x)d\mu(y) \\
&= \frac{1}{2}\int_{X\times X} f^{k-1}(x)f^{k-1}(y)(f(x)-f(y))^2\, d\mu(x)d\mu(y).
\end{aligned}$$

To iterate this argument, observe that $\mathcal{L}\mathtt{a}$ also satisfies the hypothesis of this proposition. □

3. Examples of combinatorial sequences

This section presents a list of examples of logconvex sequences using Proposition 2.1.

Example 3.1. The central binomial coefficients $\left\{ \binom{2k}{k} \right\}_k$ are infinitely logconvex.

Proof. This follows directly from Wallis' formula written in the form

$$\binom{2k}{k} = \frac{2}{\pi} \int_0^{\pi/2} (2 \sin x)^{2k} \, dx. \tag{3.1}$$

□

Example 3.2. The Catalan numbers $C_k = \frac{1}{k+1}\binom{2k}{k}$ are infinitely logconvex.

Proof. Use the integral representation

$$C_k = \frac{2}{\pi} \int_0^{\pi/2} \int_0^1 \left(4t \sin^2 x\right)^k \, dx \, dt. \tag{3.2}$$

□

Example 3.3. The generating function of the Catalan numbers C_k is

$$G(x) = \frac{2}{1 + \sqrt{1-4x}} = \sum_{k=0}^{\infty} C_k x^k. \tag{3.3}$$

Feng Qi et al [`arXiv:2005.13515v1` [mathCO] 26 May 2020] generalized the Catalan numbers and considered the function

$$G_{a,b}(x) = \frac{1}{a + \sqrt{b-x}} = \sum_{k=0}^{\infty} \mathcal{C}_k(a,b) x^k. \tag{3.4}$$

The coefficients $\mathcal{C}_k(a,b)$ admit the integral representation

$$\mathcal{C}_k(a,b) = \frac{2}{\pi} \int_0^{\infty} \frac{s^2 \, ds}{(a^2+s^2)(b+s^2)^{n+1}}. \tag{3.5}$$

Proposition 2.1 shows that, for fixed a and b, the sequence $\{\mathcal{C}_k(a,b)\}_k$ is infinitely logconvex.

Example 3.4. Let $\{F_k\}_k$ be the sequence of Fibonacci numbers. Then $\{F_{2k}/k\}$ is infinitely logconvex.

Proof. This follows from the integral representation

$$\frac{F_{2k}}{k} = \frac{1}{2} \int_0^{\pi} \left(\frac{3}{2} + \frac{\sqrt{5}}{3} \cos x \right)^{k-1} d\mu(x) \quad \text{with } d\mu(x) = \sin x \, dx. \tag{3.6}$$

□

Example 3.5. The reciprocals of the binomial coefficients $\mathtt{a}_{\mathtt{row}} = \{\binom{n}{k}^{-1}\}_k$ form an infinitely logconcave sequence. The same holds for the sequence $\mathtt{a}_{\mathtt{col}} = \{\binom{n}{k}^{-1}\}_n$.

Proof. Fix n and consider the expression $a_k = \binom{n}{k}^{-1}$. Proposition 2.1 and

$$a_k = \int_0^1 \left(\frac{x}{1-x} \right)^k d\mu(k) \quad \text{with } d\mu(x) = (n+1)(1-x)^n \, dx \tag{3.7}$$

yield the infinite logconvexity of $\mathtt{a}_{\mathtt{row}} = \{a_k\}_k$.

The second assertion follows from the representation

$$\binom{n}{k}^{-1} = \int_0^1 (n+1)(1-x)^n \, d\eta(x) \quad \text{with } d\eta(x) = \left(\frac{x}{1-x} \right)^k dx. \tag{3.8}$$

□

Example 3.6. The `derangement sequence` d_k is defined as the number of permutations in $\mathfrak{S}_k$ without fixed points. The representation of the even-indexed subsequence

$$d_{2k} = \int_0^\infty (x-1)^{2k} \, d\mu(x) \quad \text{with} \quad d\mu(x) = e^{-x} dx \tag{3.9}$$

shows that $\{d_{2k}\}_k$ is infinitely logconvex.

Example 3.7. A permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ in the symmetric group $\mathfrak{S}_n$ is called `alternating` if its entries alternately rise or descend. The `Euler number` E_n counts the number of alternating permutations in $\mathfrak{S}_n$. The integral representation

$$E_{2k} = \frac{2}{\pi} \int_0^\infty \left(\frac{2 \log x}{\pi} \right)^{2k} d\mu(x) \quad \text{with} \quad d\mu(x) = \frac{dx}{1+x^2} \tag{3.10}$$

shows that $\{E_{2k}\}_k$ is infinitely logconvex.

Example 3.8. The `large Schröder numbers` S_k count the number of paths on a $k \times k$ grid from the southwest corner $(0,0)$ to the northeast corner (k,k) using only single steps north, northeast or east that do not rise above the southwest-northeast diagonal. Proposition 2.1 and the integral representation

$$S_k = \frac{1}{2\pi} \int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{1}{x^{k+2}} \, d\mu(x) \quad \text{with} \quad d\mu(x) = \sqrt{-x^2+6x-1} \, dx \tag{3.11}$$

show that $\{S_k\}_k$ is infinitely logconvex.

Example 3.9. The `Motzkin numbers` M_k count the number of lattice paths from $(0,0)$ to (k,k), consisting of steps $(0,2)$, $(2,0)$ and $(1,1)$ subject to never rising above the diagonal $y = x$. The integral representation

$$M_{2k} = \frac{2}{\pi} \int_0^\pi (1 + 2\cos x)^{2k} \, d\mu(x) \quad \text{with } d\mu(x) = \sin^2 x \, dx \tag{3.12}$$

shows that the even-indexed Motzkin sequence $\{M_{2k}\}_k$ is infinitely logconvex.

Example 3.10. Let h_k be the number of lattice paths from $(0,0)$ to $(2k,0)$ with steps $(1,1)$, $(1,-1)$ and $(2,0)$, never falling below the x-axis and with no peaks at odd level. These numbers also count the number of sets of all tree-like polyhexes with $k+1$ hexagons. This is sequence `A002212` in OEIS. The integral representation

$$h_k = \frac{1}{2\pi}\int_1^5 x^{k-1} d\mu(x) \quad \text{with } d\mu(x) = \sqrt{(x-1)(5-x)}\,dx \tag{3.13}$$

and Proposition 2.1 show that $\{h_k\}_k$ is infinitely logconvex.

Example 3.11. Let w_k be the number of walks on a cubic lattice with k steps, starting and finishing on the xy-plane conditioned to never going below it. This is sequence `A005572` in OEIS. These numbers have the integral representation

$$w_k = \frac{1}{2\pi}\int_2^6 x^k d\mu(x) \quad \text{with } \ d\mu(x) = \sqrt{4-(4-x)^2}. \tag{3.14}$$

The usual argument shows that $\{h_k\}_k$ is infinitely logconvex.

Example 3.12. The `central Delanoy numbers` (D_k) enumerate the number of *king walks* on a $k \times k$ grid, from the $(0,0)$ corner to the upper right corner (k,k). The integral representation

$$D_k = \frac{1}{\pi}\int_{3-2\sqrt{2}}^{3+2\sqrt{2}} \frac{1}{x^{k+1}}\,d\mu(x) \quad \text{with } d\mu(x) = \frac{dx}{\sqrt{-x^2+6x-1}} \tag{3.15}$$

shows that $\{D_k\}_k$ is infinitely logconvex.

Example 3.13. The `Narayana numbers` $N(n,k)$ count the number of lattice paths from $(0,0)$ to $(2n,0)$, with k peaks, not straying below the x-axis and using northeast and southeast steps. The infinite logconvexity of the reciprocals of $N(n,k)$ follows from the integral representation

$$\frac{1}{N(n,k)} = \int_0^1\int_0^1 \left(\frac{x}{1-x}\right)^k \left(\frac{y}{1-y}\right)^{k-1} d\mu(x,y) \tag{3.16}$$

where $d\mu(x,y) = n(n+1)^2(1-x)^n(1-y)^n\,dx\,dy$.

4. A variety of examples coming from special functions

This section presents a selection of sequences related to classical special functions.

Example 4.1. The sequence of `factorials` $\{k!\}_k$ is infinitely logconvex.

Proof. Apply the representation

$$k! = \int_0^\infty x^k\,d\mu(x) \quad \text{with} \quad d\mu(x) = e^{-x}dx. \tag{4.1}$$

□

Example 4.2. The classical Eulerian `gamma` and `beta functions` are defined by integral representations

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,dt \tag{4.2}$$

and

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt. \tag{4.3}$$

Specialization of these formulae and Proposition 2.1 give infinitely logconvex sequences. Example 4.1 corresponds to the special value $\Gamma(k+1) = k!$. Another infinitely logconvex sequence arising in this manner is $\{a_k\}_k$, with

$$a_k = \frac{(2k)!}{2^{2k}k!} = \frac{1}{\sqrt{\pi}}\Gamma\left(k+\tfrac{1}{2}\right). \tag{4.4}$$

Naturally, the specialization of (4.3) gives a double-indexed logconvex sequence (symmetric in m and n)

$$B(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} = \frac{(n-1)!(m-1)!}{(n+m-1)!}. \tag{4.5}$$

Clearly, many other examples can be produced in this manner.

Example 4.3. The integral representation of the Riemann zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}\,dx}{e^x-1} \tag{4.6}$$

gives, for $k \in \mathbb{N}$,

$$\Gamma(k)\zeta(k) = \int_0^\infty x^k d\mu(x) \quad \text{with } d\mu(x) = \frac{dx}{x(e^x-1)}. \tag{4.7}$$

Proposition 2.1 shows that the sequence $\{\Gamma(k)\zeta(k)\}_k$ is infinitely logconvex.

Example 4.4. The values of the Riemann zeta function at even integers is given in terms of the `Bernoulli numbers` B_{2k} defined by the generating function

$$\coth x = \frac{1}{x}\sum_{k=0}^\infty \frac{B_{2k}}{(2k)!}(2x)^{2k}. \tag{4.8}$$

The aformentioned relation is

$$B_{2k} = \frac{(-1)^{k+1}2(2k)!}{(2\pi)^{2k}}\zeta(2k). \tag{4.9}$$

The integral representation (4.6) yields

$$\frac{B_{4k+2}}{4k+2} = \int_0^\infty 2\left(\frac{x}{2\pi}\right)^{4k+2} d\mu(x) \quad \text{with } d\mu(x) = \frac{dx}{x(e^x-1)}. \tag{4.10}$$

From here it follows that the sequence $\left\{\frac{1}{4k+2}B_{4k+2}\right\}_k$ is infinitely logconvex.

The final example of this section emerges from a multi-dimensional integral:

Example 4.5. Fix $d \in \mathbb{N}$. Then the sequence $\left\{ \frac{1}{(k+1)^d} \right\}_k$ is infinitely logconvex.

Proof. Apply the representation

$$\frac{1}{(k+1)^d} = \int_0^1 \cdots \int_0^1 (x_1 x_2 \cdots x_d)^k \, d\mu(\mathbf{x}) \tag{4.11}$$

with $d\mu(\mathbf{x}) = dx_1 dx_2 \cdots dx_d$. □

5. The motivating example

As mentioned in the Introduction, the sequence that lead the authors to the present work results from the evaluation of the quartic integral

$$N_{0,4}(a;n) = \int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{n+1}}. \tag{5.1}$$

The main result of [4] is that the expression

$$P_n(a) = \frac{1}{\pi} 2^{n+3/2} (a+1)^{n+1/2} N_{0,4}(a;n) \tag{5.2}$$

is a *polynomial* in a, of degree n, with the coefficient of a^i given by

$$d_i(n) = \sum_{k=i}^{n} 2^{k-2n} \binom{2n-2k}{n-k} \binom{n+k}{k} \binom{k}{i}. \tag{5.3}$$

Properties of these coefficients are reviewed in [10]. In particular, for fixed n, the sequence $(d_i(n))_i$ was shown to be unimodal in [1, 3, 5]. Its logconcavity was established in [9] and its 2-logconcavity appeared in [7]. The question about the infinite logconcavity of $\{d_i(n)\}_i$ remains open. The next statement follows from Proposition 2.1:

Proposition 5.1. *For fixed $r \in \mathbb{N}$, the sequence $\{P_n(r)\}_n$ is infinitely logconvex.*

Proof. Proposition 2.1 and the integral representation

$$P_n(r) = \frac{2^{3/2}\sqrt{r+1}}{\pi} \int_0^\infty \left(\frac{2(r+1)}{x^4+2rx^2+1} \right)^n d\mu(x) \tag{5.4}$$

with $d\mu(x) = \frac{dx}{x^4+2rx^2+1}$, yield the result. □

6. Chromatic polynomials

This last section discusses properties of chromatic polynomials of graphs. Recall that given an undirected graph G and x distinct colors, the number of proper colorings (adjacent vertices having distinct colors) is a polynomial in x, called the `chromatic polynomial` of G and denoted by $\kappa_G(x)$.

Examples of chromatic polynomials include

- If G is a graph with n vertices and no edges, then $\kappa_G(x) = x^n$;
- If G is a tree with n vertices, then $\kappa_G(x) = x(x-1)^{n-1}$;
- If G is the complete graph with n vertices, then
$$\kappa_G(x) = x(x-1)\cdots(x-n+1).$$

In these examples, the chromatic polynomials have only real roots. The logconcavity of the coefficients follows from a work of P. Bränden [6].

Other examples of chromatic polynomials include

- For a cycle G with n vertices, $\kappa_G(x) = (x-1)^n + (-1)^n(x-1)$;
- If G is the bipartite graph $K_{n,m}$, then
$$\kappa_G(x) = \sum_{j=0}^{m} S(m,j)(x)_j(x-j)^n,$$
where $S(m,k)$ is the `Stirling number of the second kind` and $(x)_k = x(x-1)\cdots(x-k+1)$ is the falling factorial.
- If G is the `cyclic ladder` graph with $2n$ vertices, then

$$\kappa_G(x) = (x^2-3x+3)^n - (1-x)^{n+1} - (1-x)(3-x)^n + (x^2-3x+1). \tag{6.1}$$

- If G is the `signed book graph` $B(m,n)$, then

$$\kappa_G(x) = (x-1)^m x^{-n}\left((x-1)^m + (-1)^m\right)^n. \tag{6.2}$$

These examples, as well as many more from the long list given by Birkhoff and Lewis [2], have been tested to be infinitely logconcave.

J. Huh [8] proved:

Theorem 6.1. *The absolute values of the coefficients of a chromatic polynomial $\kappa_G(x)$ are logconcave.*

The authors will analyze chromatic polynomials by the methods presented here. In the meantime, based on some experimental evidence, we invite the reader to:

Conjecture 6.2. *The absolute values of any chromatic polynomial are infinitely logconcave.*

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