A CONDENSED CONDENSATION PROOF OF A DETERMINANT EVALUATION CONJECTURED BY Greg KUPERBERG AND Jim PROPP

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Greg Kuperberg and Jim Propp [P] have conjectured the following determinant identity:

$$\det\left[\binom{i+j}{i}\binom{2n-i-j}{n-i}_{0\le i,j\le n}\right] = \frac{(2n+1)!^{n+1}}{(2n+1)!!} \quad ,$$

where $a!! := 0!\cdot 1!\cdot 2!\cdots a!$, and $a! := 1\cdot 2\cdots a$.

This is the special case $(m=n, a=b=0)$ of

$$\det\left[\binom{i+j+a+b}{i+a}\binom{2n-i-j-a-b}{n-i-a}_{0\le i,j\le m}\right] =$$

$$\frac{(a+b)!(2n+1)!^{m+1}(2n-m)!!m!!(m+a+b)!!(2n-m-a-b)!!a!!b!!(n-m-a-1)!!(n-m-b-1)!!}{a!b!(2n+1)!!(n-a)!!(n-b)!!(m+a)!!(m+b)!!(a+b)!!(2n-2m-a-b-1)!!} \quad , \qquad (Rabbit)$$

which follows immediately from Dodgson's[D] rule for evaluating determinants: (For any $n\times n$ matrix A, let $A_r(k,l)$ be the $r\times r$ connected submatrix whose upper leftmost corner is the entry $a_{k,l}$,)

$$\det A \;=\; \frac{\det A_{n-1}(1,1)\det A_{n-1}(2,2)-\det A_{n-1}(1,2)\det A_{n-1}(2,1)}{\det A_{n-2}(2,2)} \quad . \qquad (Lewis)$$

Indeed, let the left and right sides of $(Rabbit)$ be $L_m(a,b)$ and $R_m(a,b)$ respectively. Dodgson's rule immediately implies that the recurrence:

$$X_m(a,b) = \frac{X_{m-1}(a,b)X_{m-1}(a+1,b+1) - X_{m-1}(a+1,b)X_{m-1}(a,b+1)}{X_{m-2}(a+1,b+1)} \quad ,$$

holds with $X=L$. Since $L_m(a,b)=R_m(a,b)$ for $m=0,1$ (check!), and the recurrence also holds with $X=R$ (check![2]), it follows by induction that $L_m(a,b)=R_m(a,b)$ for *all* m. $\Box$.

The present proof is in the spirit of [Z1]. Another proof can be found in [A1]. The same method yields a q-analog of $(Rabbit)$, that can be found in [A2]. A beautiful combinatorial proof of $(Lewis)$ can be found in [Z2]. An alternative proof of $(Rabbit)$ is given in [K].

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[2] Divide both sides by the left, then use r!!/(r-1)!!=r! whenever possible, and then r!/(r-1)!=r whenever possible, reducing it to a completely routine polynomial identity. The small Maple package `rabbit` obtainable from our Home Pages, performs these steps mechanically.

References

[A1] T. Amdeberhan, *A WZ proof of a determinant evaluation conjectured by Kuperberg and Propp*, exclusively published in Amdeberhan's Home Page `http://www.math.temple.edu/~tewodros`.

[A2] T. Amdeberhan, *A q-generalization of a determinant evaluation conjectured by Kuperberg and Propp*, exclusively published in Amdeberhan's Home Page `http://www.math.temple.edu/~tewodros`.

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[K] C. Krattenthaler, *E-mail message to T. Amdeberhan*, dated 9 Aug. 1996, $19:22:27+0100$ (MET). By kind permission of Krattenthaler, it can be gleaned at the first author's Home Page `http://www.math.temple.edu/~tewodros`.

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[Z1] D. Zeilberger, *Reverend Charles to the aid of Major Percy and Fields Medalist Enrico*, Amer. Math. Monthly **103**(1996), 501-502.

[Z2] D. Zeilberger, *Dodgson's determinant-evaluation rule proved by Two-Timing Men and Women*, to appear in the Elec. J. of Combinatorics [Wilf Festschrifft volume]. It can be downloaded from Zeilberger's Home Page: `http://www.math.temple.edu/~zeilberg`.

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