ON INJECTIVITY OF COMBINATORIAL RADON TRANSFORM OF ORDER FIVE

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Abstract. In the present work, we give a proof of the injectivity of the combinatorial radon transform of order five.

The problem of determining members of a set by their sums of a fixed order was posed by Leo Moser and partially settled by Ewell, Fraenkel, Gordon, Selfridge, and Straus. Following the notation of [BL], the general problem can be stated in the following way.

For any given $(k, n) \in \mathbb{Z} \times \mathbb{Z}$, with $2 \leq k \leq n$, we choose arbitrarily an n-set $X_n = \{x_1, x_2, ..., x_n\}$ then form the set $W_n^k(X_n) = \{\sigma_i\}$ of all sums of k distinct elements of X_n and ask:
Does there exist an n-set X'_n different from X_n giving rise to the same set of sums as does X_n? More formally, we can describe the problem as follows:
Define a mapping W_n^k from the set $\{X_n\}$ of all n-sets to the set of all $\binom{n}{k}$-sets by the rule:

$$W_n^k(\{x_1, x_2, \dots, x_n\}) = \{x_{i_1} + x_{i_2} + \cdots + x_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

and try to determine whether W_n^k is one-to-one.

Definition: W_n^k is called a combinatorial radon transform of order k.

It is known **[E], [FGS], [SS]** that W_n^k is injective if

$$A(n, k, s) = \sum_{i=0}^{k-1} (-1)^i \binom{n}{i} (k-i)^{s-1} \neq 0 \tag{1}$$

for each s in $\{1, 2, \dots, n\}$.

Remarks: The following results are also known.

(1) if $k = 2$, W_n^2 is injective for all n which are *not* a power of 2. W_n^2 is *not* injective if n is a power of 2 **[SS]**.
(2) if $k = 3$, W_n^3 is injective for all $n \geq 3$ and $n \neq 3, 6, 27$, and 486. W_n^k is not injective if $n = 3, 6, 27$, **[BL], [EZ]** or 486 **[BL]**.

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(3) If $k = 4$, W_n^4 is injective for all $n \geq 4$ and $n \neq 4$ and 8. W_n^k is not injective if $n = 4$, or 8 **[E], [EZ]**. Here we would like to point out that while $A(12, 4, 6) = 0$, John Ewell proved W_{12}^4 is injective, thereby showing that condition (1), though necessary, is *not* sufficient.

In this paper, we settle the problem for the combinatorial radon transform of order five.

In the case $k = 5$, condition (1) reduces to a polynomial in $n, 2^s, 3^s, 5^s$, and it can be written as: W_n^5 is injective if

$$\begin{aligned} A(n,5,s) &= n^4 - (2^{s+1}+6)n^3 + (4\cdot 3^s + 3\cdot 2^{s+1} + 11)n^2 \\ &\quad - (4\cdot 3^s + 3\cdot 2^{2s+1} + 2^{s+2} + 6)n + 24\cdot 5^{s-1} \neq 0 \end{aligned} \tag{2}$$

for every $s \in \{1, 2, \ldots, n\}$.

Consider the function

$$B(n,s) = n^4 + a_3n^3 + a_2n^2 + a_1n + a_0;$$

where

$a_3 = -2(2^s + 3)$ $\qquad$ $a_2 = 4\cdot 3^s + 3\cdot 2^{s+1} + 11$

$a_1 = -2(2\cdot 3^s + 3\cdot 4^s + 2^{s+1} + 3)$ $\qquad$ $a_0 = 2^3\cdot 3\cdot 5^{s-1}$

for integers $1 \leq s \leq n$.

Let n be an integral solution of

$$B(n,s) = 0. \tag{3}$$

Note: n must have the form

$$n = 2^\alpha \cdot 3^\beta \cdot 5^\gamma \tag{4}$$

for $\alpha = 0, 1, 2, 3; \beta = 0, 1; \gamma = 0, 1, 2, 3, \ldots, s-1$.

So, throughout this investigation we assume that n has the form (4).
Dividing (3) by n we get:

$$\tilde{B}(n,s) = n^3 + a_3n^2 + a_2n + a_1 + \tilde{a}_0 = 0 \tag{5}$$

where $\tilde{a}_0 = 2^{3-\alpha}\cdot 3^{1-\beta}\cdot 5^{s-1-\gamma}$.

Observations:

(1) α cannot be 3: if $\alpha = 3$, then $\tilde{B}(n,s) \not\equiv 0 \pmod 2$ since $2 \nmid \tilde{a}_0$, but 2 divides the rest of the terms.

(2) α cannot be 2: if $\alpha = 2, \beta = 1$ then similarly $\tilde{B}(n,s) \not\equiv 0 \pmod 8$ (in fact $\tilde{B}(n,s) \equiv 4 \pmod 8$).
If $\alpha = 2, \beta = 0$, then likewise $\tilde{B}(n,s) \not\equiv 0 \pmod{16}$.

(3) $\alpha = 1, \beta = 1$ is not possible: $\tilde{B}(n,s) \not\equiv 0 \pmod 8$.

Hence we gather that n takes one of the forms:

$$n = 2 \cdot 5^\gamma \; or \; n = 3 \cdot 5^\gamma \; or \; n = 5^\gamma. \tag{6}$$

On the other hand, if $n > 2(2^s + 3)$, then

$$n^4 + a_3 n^3 = n^3(n - 2(2^s + 3)) > 0,$$

and moreover

$$\begin{aligned} a_2 n^2 + a_1 n &= n\left\{(4 \cdot 3^s + 6 \cdot 2^s + 11)n - (4 \cdot 3^s + 6 \cdot 4^s + 2^{s+2} + 6)\right\} \\ &> n\left\{(4 \cdot 3^s + 6 \cdot 2^s + 11)2^{s+1} - (4 \cdot 3^s + 6 \cdot 4^s + 2^{s+2} + 6)\right\} \\ &> n\{(4 \cdot 3^s \cdot 2^{s+1} + 6 \cdot 4^s \cdot 2 + 5 \cdot 2^{s+1} + 6 \cdot 2^{s+1}) \\ &\quad - (4 \cdot 3^s + 6 \cdot 4^s + 2 \cdot 2^{s+2} + 6)\} \\ &> 0. \end{aligned}$$

Hence, $B(n, s) > 0$ if $n > 2(2^s + 3)$.
Note however that if $\gamma \geq \frac{1}{2}s + 1$, then $n > 2(2^s + 3)$. This implies for such γ, the equation $B(n, s) = 0$ has no integral solution n. Therefore, in the sequel it suffices to assume that $\gamma < \frac{1}{2}s + 1$.

Notation: Let $ord_p x$ denote the exponent of a prime p in the prime factorization of x.

Lemma:. *If $5^m \leq s < 5^{m+1}$ for any fixed $m \geq 0$, then $\mu := ord_5 \tilde{a}_1(s) \leq m + 2$.*

Proof. Using the *binomial theorem*

$$(5 - x)^s = (-1)^s x^s + (-1)^{s-1} 5 \cdot s \cdot x^{s-1} + \dots \tag{7}$$

Let $\tilde{a}_1 := 2 \cdot 3^s + 3 \cdot 4^s + 2 \cdot 2^s + 3$. Then $\tilde{a}_1 = 3(4^s + 1) + 2(3^s + 2^s)$, and in light of (7) we can rewrite $\tilde{a}_1$ as

$$\tilde{a}_1 = (3 \cdot 5 \cdot s + 2 \cdot 5 \cdot s \cdot 2^{s-1}) + \dots \;,$$

for s odd.
Writing s in the form

$$s = k_m 5^m + k_{m-1} 5^{m-1} + \dots + k_1 5 + k_0; \; 0 \leq k_i \leq 4, \; for \; all \; i,$$

define $j := min\{i | k_i \neq 0\}$. Then $s = k_m 5^s + \dots + k_j 5^j$. Note that $k_m \geq 1, k_j \geq 1$.

Therefore, for s odd,

$$\begin{aligned}\tilde{a}_1 &= 5(3\cdot k_m\cdot 5^m + 2\cdot k_m\cdot 5^m\cdot 2^{s-1}) + ... + 5(3\cdot k_j\cdot 5^j + 2\cdot k_j\cdot 5^j\cdot 2^{s-1}) + \dots \\ &= 5^{m+1}(3+2^s)k_m + \dots + 5^{j+1}(3+2^s)k_j + \dots\end{aligned}$$

Hence

$$ord_5\tilde{a}_1(s) \le j+2 \le m+2.$$

For s even,

$$x^s + (5-x)^s = 2x^s - 5sx^{s-1} + 5^2\binom{s}{1}x^{s-2} - + \dots$$

Thus,

$$\tilde{a}_1 = 2(3+2^{s+1}) - 5s(3+2^s) + 5^2\binom{s}{1}(3+2^s) - + \dots$$

Writing s in the form:

$$s = k_m 5^s + ... + k_j 5^j,\ j\ as\ in\ above,$$

we see that

$$\tilde{a}_1 = 2(3+2^{s+1}) - k_m 5^{m+1}(3+2^s) - \dots - k_j 5^{j+1}(3+2^s) + \dots$$

But $5 \nmid (3+2^s)$, while $5 \mid (3+2^{s+1})$ as s is even. Thus $\tilde{a}_1 = 2^{s+1} - (3+2^s)\cdot 5(s-2) + \dots$ is at most divisible by 5^{m+2} since $5^m \le s < 5^{m+1}$. Hence if s is even and $5^m \le s < 5^{m+1}$, then $Ord_5\tilde{a}_1(s) \le m+2$.$\blacksquare$

Now,

(1) Suppose that $\gamma \ge m+1$. Then $m+1 \le \gamma < \frac{1}{2}(s+1)$.
   (i) if $s-1-\gamma \ge m+1$, then $\tilde{B}(n,s) \not\equiv 0(mod\ 5^{m+1})$ since $5^{m+1} \nmid a_1$ by the lemma above.
   (ii) if $s-1-\gamma < \mu$, then $B(n,s) \not\equiv 0(mod\ 5^{\mu})$ as $5^{\mu} \nmid \tilde{a}_0$.
   (iii) if $s-1-\gamma = \mu$, then $\gamma = s-1-\mu \ge s-1-m$ by the lemma above. But then $s-1-m < \frac{1}{2}s+1$.

   Therefore,

$$5^{m-2} \le s < 2m+4.$$

   Hence $m \le 3$.

(2) Suppose that $m-2 \le \gamma \le m$.
   If $m \ge 4$, then $a_0(s) = 24\cdot 5^{s-1} > -a_1 n - a_3 n^3$ for n in one of the above forms. We then conclude that $B(n,s) > 0$, that is, equation (3) has no integral solution unless $m \le 3$.

Conclusion: In all cases, $m \leq 3$. This shows that it remains to verify whether n in the form (4) is a solution of equation (3) for $0 \leq \gamma < \frac{1}{2}s + 1 \leq \frac{1}{2}5^2 + 1 \leq 14$. (Recall that for $m \leq 3$, we also have $1 \leq s \leq 24$.) That is, we simply test if

$$B(n,s) = 0 \ for \ 0 \leq \gamma \leq 14, \ 1 \leq s \leq 24. \tag{8}$$

We carried out this test using **Maple***, and found that (8) is true only if $n = 2, 3, 4, 5, \ or \ 10$.

Thus we have proved the following theorem:

Theorem. *Let n and s be positive integers such that $s \leq n$. Then*

$$B(n,s) = 0$$

only if $n = 2, 3, 4, 5, \ or \ 10$.

Corollary. *The combinatorial radon transform of order five is injective for all $n \geq 5$ and $n \neq 5, \ and \ 10$.*

Note: W_5^5 is clearly noninjective and W_{10}^5 is not injective since

$$X := \{0^1, 5^6, 10^3\} \neq Y := \{2^3, 7^6, 12^1\}$$

but

$$W_{10}^5(X) = W_{10}^5(Y)$$

[EZ].

*A short Maple program that carries out the test is available in the **WWW** under **http://www.math.temple.edu˜[melkamu,tewodros]**.

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