Positivity of Szegö's Rational Function

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Abstract

We consider the problem of deciding whether a given rational function has a power series expansion with all its coefficients positive. Introducing an elementary transformation that preserves such positivity we are able to provide an elementary proof for the positivity of Szegö's function

$$\frac{1}{(1-x)(1-y)+(1-y)(1-z)+(1-z)(1-x)}$$

which has been at the historical root of this subject starting with Szegö. We then demonstrate how to apply the transformation to prove a 4-dimensional generalization of the above function, continue with a brief discussion concerning the optimality of this transformation, and close with an elementary proof of a conjecture raised by Kauers.

1 Introduction

In 1930 H. Lewy and K. Friedrichs conjectured that the coefficients $a(k, m, n)$ in the expansion

$$\frac{1}{(1-x)(1-y)+(1-y)(1-z)+(1-z)(1-x)} = \sum_{k,m,n\geqslant 0} a(k,m,n)\, x^k\, y^m\, z^n \tag{1.1}$$

are all positive. This was proved shortly after by G. Szegö employing heavy machinery in [Szegö, 1933], but he remarks himself "die angewendeten Hilfsmittel stehen allerdings in keinem Verhältnis zu der Einfachheit des Satzes"[1.1]. Motivated by these words, T. Kaluza gave an elementary but technically difficult proof that was published in the very same journal [Kaluza, 1933]. R. Askey and G. Gasper also proved the above positivity in [Askey and Gasper, 1972] using some of Szegö's observations but avoiding the use of Bessel functions in favour of Legendre polynomials. The problem has also been considered in the recent paper [Kauers, 2007] by M. Kauers from the viewpoint of computer algebra, and Kauers establishes the result under the constraint that $k \leqslant 16$ by finding appropriate recurrences. We provide an elementary proof of Szegö's result with the main ingredient being a simple positivity preserving operation in the spirit of [Gillis et al., 1983], whence we reduce the positivity of the coefficients $a(k, m, n)$ to the positivity of another rational function that is easier to handle. While our proof is indeed elementary, to check that the latter rational function is *positive*[1.2] is most conveniently done with the aid of computer algebra.

2 A Positivity Preserving Operation

The following elementary proposition is related to the positivity preserving operations given by J. Gillis, B. Reznick and D. Zeilberger in [Gillis et al., 1983, Proposition 2]. In fact, it would be a special case if $1/(1-x-x\,y)$ had positive coefficients.

Proposition 2.1. *Suppose that the polynomial $p(x, y, z)$ is linear in x. If $1/p(x, y, z)$ has positive coefficients then so does*

$$\frac{1}{p(x,y,z)-x\,p(0,y,z)}.$$

1.1. the used tools, however, are in no proportion to the simplicity of the statement

1.2. although a bit sloppy, we will stick to calling a rational function positive meaning that all its Taylor coefficients are positive

Proof. Write $p(x, y, z) = a(y, z) - x\, b(y, z)$. Since

$$\frac{1}{a(y, z) - x\, b(y, z)} = \sum_{n \geqslant 0} \frac{b(y, z)^n}{a(y, z)^{n+1}}\, x^n$$

has positive coefficients so has b^n / a^{n+1}. The quotient

$$\frac{b(y, z)^n}{a(y, z)^{n+1}} \frac{x^n}{(1-x)^{n+1}}$$

has nonnegative coefficients, and for $n = 0$ they are all positive. This finally implies the positivity of

$$\sum_{n \geqslant 0} \frac{(x\, b(y, z))^n}{((1-x) a(y, z))^{n+1}} = \frac{1}{(1-x)\, a(y, z) - x\, b(y, z)}. \qquad \square$$

Reading proposition 2.1 the other way round, in order to establish the positivity of $1/p(x, y, z)$ it is sufficient to do so for $(p(x, y, z) + x\, p(0, y, z))^{-1}$. Applying this repeatedly to each variable, provided that p is linear in each of them, shows that it is further sufficient to prove the positivity of

$$\left(\begin{array}{c} p(x, y, z) + x\, p(0, y, z) + y\, p(x, 0, z) + z\, p(x, y, 0) \\ + x\, y\, p(0, 0, z) + y\, z\, p(x, 0, 0) + z\, x\, p(0, y, 0) + x\, y\, z\, p(0, 0, 0) \end{array}\right)^{-1}.$$

This is again a symmetric rational function whenever $1/p(x, y, z)$ is.

Remark 2.2. Of course, proposition 2.1 generalizes to higher dimensions. It is also readily verified that under the given hypothesis the rational function

$$\frac{1}{p(x, y, z) - \lambda x\, p(0, y, z)}$$

has positive coefficients for any $\lambda \geqslant 0$. We will make use of this later.

3 Szegö's Rational Function

Theorem 3.1. *Szegö's rational function*

$$f(x, y, z) \triangleq \frac{1}{1 - 2\,(x + y + z) + 3\,(x\, y + y\, z + z\, x)}$$

is positive.

Remark 3.2. Note that up to rescaling this is the rational function from (1.1), namely

$$\frac{1}{3} f\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right) = \frac{1}{(1-x)(1-y) + (1-y)(1-z) + (1-z)(1-x)}.$$

Proof. The denominator of f is linear in all the variables x, y, z, so we can apply our inverted positivity preserving operation repeatedly to x, y, z. Doing so we obtain

$$g(x, y, z) \triangleq \frac{1}{1 - (x + y + z) + 4\, x\, y\, z}.$$

Positivity of $g(x, y, z)$ now implies positivity of $f(x, y, z)$.

The positivity of g, however, is well-known, and several short proofs have been given in the literature (not so, to our knowledge, for f). One possibility is to verify by differentiation that the coefficients $b(k, m, n)$ of g satisfy the following recurrence, first observed by J. Gillis and J. Kleeman [Gillis and Kleeman, 1979],

$$(1 + n)\, b(k + 1, m + 1, n + 1) = 2\,(n + m - k)\, b(k + 1, m, n) + (1 + n - m + k)\, b(k + 1, m + 1, n),$$

which together with the initial $b(0, 0, 0) = 1$ proves positivity of the $b(k, m, n)$ by induction. $\square$

Kauers describes in [Kauers, 2007] how to automatically find positivity proving recurrences with computer algebra, and also remarks that no such first-order recurrence with linear coefficients exists for Szegö's f. Note that once such a recurrence is guessed, for instance using a computer, proving it is just a matter of verifying that the generating function solves the corresponding differential equation.
Another simple proof of the positivity of g based on MacMahon's master theorem is given by M. Ismail and M. Tamhankar in [Ismail and Tamhankar, 1979]. We will discuss this theorem in section 5. The reason for doing so is that we discovered the transformation presented in proposition 2.1 by applying MacMahon's master theorem, whence it is possible to just *see* its impact.

4 A 4-Dimensional Generalization

Following [Szegö, 1933, §3] we define

$$q_n(t) = \prod_{k=1}^{n} (t - x_k),$$

and observe that one can recover Szegö's function as

$$\frac{1}{q_3'(1)} = \frac{1}{(1-x_1)(1-x_2) + (1-x_2)(1-x_3) + (1-x_3)(1-x_1)}.$$

Szegö proves that $1/q_n'(1)$ as a rational function in x_1, ..., x_n has positive Taylor coefficients for all $n \geqslant 2$, and remarks that the essential difficulty lies in the cases $n = 3$ and $n = 4$. While our previous discussion covers $n = 3$, we now want to briefly demonstrate how to use proposition 2.1 to also establish the case $n = 4$ in an elementary way.

Theorem 4.1. *The rational function*

$$\frac{1}{q_4'(1)} = \frac{1}{\sum_{i<j<k} (1-x_i)(1-x_j)(1-x_k)},$$

where $i, j, k = 1, 2, 3, 4$, is positive.

Proof. Expanding the denominator of $1/q_4'(1)$ and rescaling produces the rational function

$$\frac{1}{1 - 3\sum_i x_i + 8\sum_{i<j} x_i x_j - 16\sum_{i<j<k} x_i x_j x_k}.$$

Applying the inverted positivity preserving operation from proposition 2.1 repeatedly to x_1, x_2, x_3, x_4 twice, noting that order doesn't matter, (or with $\lambda = 2$ as in remark 2.2) we find that it suffices to establish positivity of

$$\frac{1}{1 - \sum_i x_i + 4\sum_{i<j<k} x_i x_j x_k - 16\, x_1 x_2 x_3 x_4}.$$

This again is a well-known result. In particular, Gillis, Reznick and Zeilberger demonstrate in [Gillis et al., 1983] how a single application of their elementary methods can be used to deduce the desired positivity. □

For other possible generalizations of Szegö's function the interested reader is referred to [Askey and Gasper, 1972].

5 MacMahon's Master Theorem

The following is a celebrated result of P. A. MacMahon published in [MacMahon, 1915], and coined by himself as "a master theorem in the Theory of Permutations".

Theorem 5.1. (MacMahon, 1915) *Let R be a commutative ring, $A \in R^{n\times n}$ a matrix, and $x = (x_1, \ldots, x_n)$ commuting indeterminants. For every multi-index $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geqslant 0}$*

$$[x^m] \prod_{i=1}^{n} \left(\sum_{j=1}^{n} A_{i,j}\, x_i \right)^{m_i} = [x^m] \det\left(I_n - A \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \right)^{-1},$$

where $[x^m]$ denotes the coefficient of $x_1^{m_1} \cdots x_n^{m_n}$ in the expansion of what follows.

We find, preferably by using computer algebra, that Szegö's function $f(x, y, z)$ can be expressed as

$$\frac{1}{1 - 2\,(x + y + z) + 3\,(x\,y + y\,z + z\,x)} = \det\left(I_3 - \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \right)^{-1}.$$

MacMahon's theorem 5.1 now asserts that the coefficient $a(k, m, n)$ of $x^k\, y^m\, z^n$ in this expansion is equal to the coefficient of $x^k\, y^m\, z^n$ in

$$(2x - y - z)^k\,(-x + 2y - z)^m\,(-x - y + 2z)^n.$$

Using the binomial theorem this product is equal to

$$\sum_{r,s,t} \binom{k}{r} \binom{m}{s} \binom{n}{t} x^{k-r}\, y^{m-s}\, z^{n-t}\,(x - y - z)^r\,(-x + y - z)^s\,(-x - y + z)^t,$$

which shows that in order to establish positivity of the $a(k, m, n)$ it is sufficient to prove positivity of the coefficient of $x^r\, y^s\, z^t$ in

$$(x - y - z)^r\,(-x + y - z)^s\,(-x - y + z)^t.$$

By applying MacMahon's master theorem 5.1 backwards we find that

$$\det\left(I_3 - \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x & & \\ & y & \\ & & z \end{pmatrix} \right)^{-1} = \frac{1}{1 - (x + y + z) + 4\,x\,y\,z},$$

which once more reduces positivity of f to the positivity of g.
Increasing a diagonal element of the matrix A that appears in MacMahon's theorem by 1 is equivalent to applying the positivity preserving operation given in proposition 2.1 to the corresponding rational function. The reason for mentioning MacMahon's theorem here, is that one can just *see* that starting with Szegö's function f we can reduce to g by decreasing all diagonal elements by 1.

6 Optimality

Kauers states that "it is easy to show that there can be no algorithm which for a given multivariate rational function decides whether all its series coefficients are positive", see [Kauers, 2007], whence we focus here on the reciprocals of certain symmetric polynomials. In the 3-dimensional case we have the 4 elementary symmetric polynomials

$$1, \quad x + y + z, \quad x\,y + y\,z + z\,x, \quad x\,y\,z,$$

and if we further require that every variable appears at most linearly, the most general normalized candidate for positivity is

$$h_{a,b}(x, y, z) = \frac{1}{1 - (x + y + z) + a\,(x\,y + y\,z + z\,x) + b\,x\,y\,z}.$$

We ask for conditions on a, b such that $h_{a,b}$ has positive coefficients. First, we note that positivity of some $h_{a,b}$ implies positivity of $h_{a',b'}$ whenever $a' < a$ and $b' < b$. This is a consequence of the following general fact.

Proposition 6.1. *Let $1/p(x_1, \ldots, x_r)$ be a positive rational function, and $q(x_1, \ldots, x_r)$ any polynomial with non-negative coefficients. Then the rational function*

$$\frac{1}{p-q}$$

is positive provided that it has no pole at the origin.

Proof. This follows from the geometric summation

$$\frac{1}{p-q} = \frac{1}{p} \sum_{n \geqslant 0} \left(\frac{q}{p}\right)^n. \qquad \square$$

Positivity of Szegö's function $f(x, y, z) = h_{3/4,0}(2x, 2y, 2z)$ now shows that if $b = 0$ then $a \leqslant 3/4$ is sufficient for positivity, whereas positivity of $g = h_{0,4}$ implies that if $a = 0$ then $b \leqslant 4$ suffices. Are these conditions optimal, that is are they necessary?

Proposition 6.2. *Suppose that the rational function $h_{0,b}(x, y, z)$ is positive. Then $b \leqslant 4$.*

Proof. We expand $h_{0,b}(x, y, z)$ as

$$h_{0,b}(x, y, z) = \frac{1}{1 - x - y - z + b\,x\,y\,z} = \sum_{n \geqslant 0} \frac{(1 - b\,x\,y)^n}{(1 - x - y)^{n+1}} z^n.$$

Using

$$\frac{1}{(1 - x - y)^{n+1}} = \sum_{m \geqslant 0} \binom{n+m}{n} (x + y)^m,$$

we conclude that the coefficient of $x\,y^n\,z^n$ in $h_{0,b}(x, y, z)$ is given by

$$(n+1)\binom{2n+1}{n} - b\,n\binom{2n-1}{n}.$$

Positivity of $h_{0,b}$ then implies that

$$b < \frac{2\,(2n+1)}{n}$$

for all integers $n > 0$. $\square$

We believe, supported by numerical evidence, that in fact both conditions are not only sufficient but also necessary.

Conjecture 6.3. *Suppose that the rational function $h_{a,0}(x, y, z)$ is positive. Then $a \leqslant 3/4$.*

Remark 6.4. As was the case in the proof of proposition 6.2 it seems that $a \leqslant 3/4$ can be deduced from just considering the coefficients of the form $x\,y^n\,z^n$ in the expansion of $h_{a,0}$.

The (conjectured) optimality of Szegö's function f is quite remarkable in this context since proposition 2.1 allows us to only conclude its positivity from the positivity of g but not vice versa. Yet proposition 2.1 seems to still provide us with an optimal result.
More seems to be true. By remark 2.2 positivity of some $h_{a,b}$ implies positivity of $h_{a',b'}$ where

$$a' = \frac{a + 2\lambda + \lambda^2}{(1+\lambda)^2}, \quad b' = \frac{b - 3\lambda a - 3\lambda^2 - \lambda^3}{(1+\lambda)^3},$$

and $\lambda \geqslant 0$. We conjecture that these parameters a', b' are optimal (in the sense that increasing either of them will destroy positivity) whenever a, b are optimal. In particular, for the discussed case $a = 0$ and $b = 4$ this reads as follows.

Conjecture 6.5. *Let $0 \leqslant a \leqslant 1$. The rational function $h_{a,b}$ has positive coefficients if and only if*

$$b \leqslant 2 - 3a + 2(1-a)^{3/2}.$$

Remark 6.6. Note that we proved the if-part of this conjecture for $a<1$. For $a=1$ we get

$$h_{1,-1}(x,y,z)=\frac{1}{(1-x)(1-y)(1-z)}$$

which is obviously positive.

Example 6.7. We illustrate this conjecture for $a=1/2$. While we know that $h_{a,b}$ has positive coefficients for

$$b\leqslant\frac{1}{2}+\frac{1}{\sqrt{2}}\approx 1.207,$$

conjecture 6.5 claims that this is optimal. Indeed we find that for instance the Taylor coefficients up to order 20 are positive only if

$$b<1.212.$$

The region of positivity as stated in conjecture 6.5 is shown in figure 6.1 with the points corresponding to Szegö's function f and g marked.

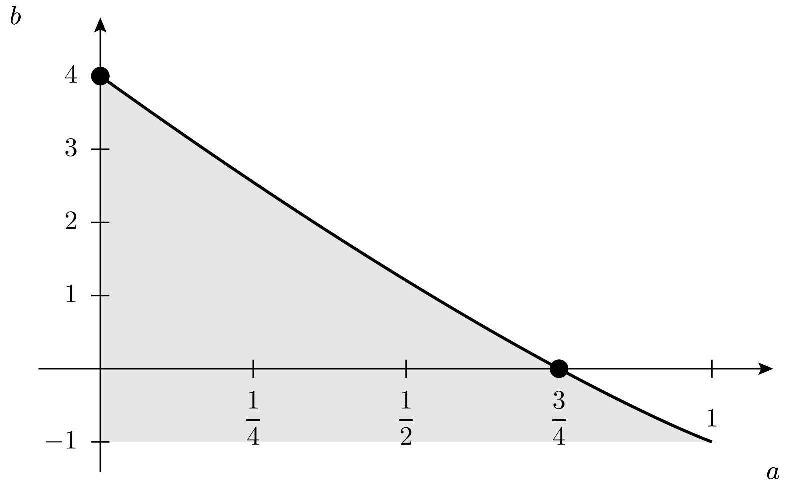

Figure 6.1. Region of Positivity of $h_{a,b}$

To further complete the picture, we show that $h_{a,b}$ can't have positive coefficients for $a>1$ regardless of b.

Proposition 6.8. *Suppose that $h_{a,b}(x,y,z)$ is positive. Then $a\leqslant 1$.*

Proof. We expand $h_{a,b}(x,y,0)$ as

$$h_{a,b}(x,y,0)=\frac{1}{1-x-y+a\,x\,y}=\sum_{n\geqslant 0}\frac{(1-a\,x)^n}{(1-x)^{n+1}}\,y^n,$$

and as in proposition 6.2 deduce that the coefficient of $x\,y^n$ in $h_{a,b}(x,y,z)$ is given by $n+1-n\,a$. Positivity of $h_{a,b}$ thus implies that

$$a<\frac{n+1}{n}$$

for all positive integers n. □

What happens for $a<0$? The coefficient of $x\,y\,z$ in the expansion of $h_{a,b}(x,y,z)$ is $6(1-a)-b$. Thus conjecture 6.5 does not hold whenever $a\leqslant a_0$, where $a_0\approx-1.81451$ is the unique real solution to

$$2-3a_0+2(1-a_0)^{3/2}=6(1-a_0).$$

Numerical evidence suggests that for $a\leqslant a_0$ the condition $6(1-a)-b>0$ is also sufficient for positivity, which we express in the following conjecture that attempts to complete the picture provided by conjecture 6.5.

Conjecture 6.9. *The rational function $h_{a,b}$ has positive coefficients if and only if*

$$\begin{cases} b < 6(1-a) \\ b \leqslant 2 - 3a + 2(1-a)^{3/2} \\ a \leqslant 1 \end{cases}.$$

The following graphic 6.2 shows the region defined by the restrictions given in conjecture 6.9.

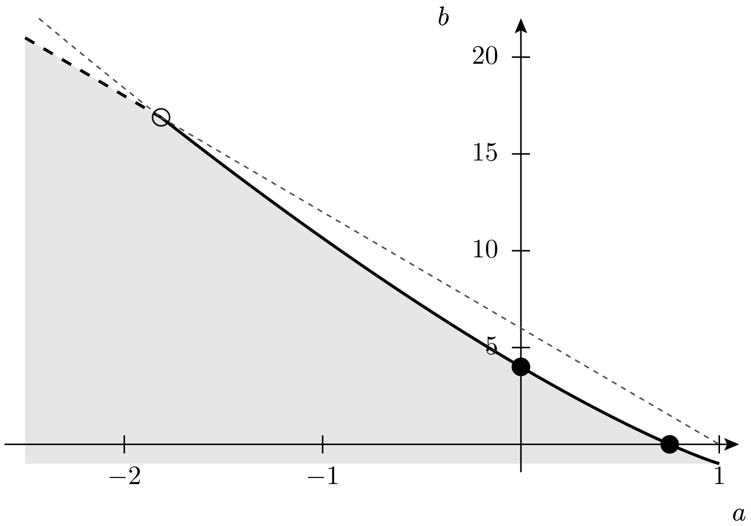

Figure 6.2. Extended Region of Positivity of $h_{a,b}$

7 On a Conjecture of Kauers

The following was conjectured to be true in [Kauers, 2007].

Theorem 7.1. *The rational function*

$$\frac{1}{1-(x+y+z)+\frac{1}{4}(x^2+y^2+z^2)}$$

is positive.

Proof. Clearly,

$$\frac{1}{(1-x)^2}$$

has positive coefficents, and thus has

$$\frac{1}{(1-\frac{x+y+z}{2})^2} = \frac{1}{1-(x+y+z)+\frac{1}{2}(x\,y+y\,z+z\,x)+\frac{1}{4}(x^2+y^2+z^2)}.$$

Proposition 6.1 now implies positivity of the function considered by Kauers. □

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