A Q-ANALOGUE DETERMINANT EVALUATION POSED BY G. KUPERBERG

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Greg Kuperberg [K] posed the following "q-analogue" determinant evaluation:

$$det\left[\left(q^{-ij}\binom{i+j}{i}_q\binom{2n-i-j}{n-i}_q\right)_{0\le i,j\le n}\right]. \tag{1}$$

Here, we propose and prove a formula for computing (1) in some generality.

Let $M_n^q := [(q^{-ij}\binom{i+j}{i}_q\binom{2n-i-j}{n-i}_q)_{0\le i,j\le n}]$ and define the matrices:

$$M_{n,m}^q(a,b) := \left[\left(q^{-(a+i)(b+j)}\binom{i+j+a+b}{i+a}_q\binom{2n-i-j-a-b}{n-i-a}_q\right)_{0\le i,j\le m}\right].$$

PROPOSITION: For the submatrices of M_n^q it holds that,

$$detM_{n,m}^q(a,b) = \prod_{i=0}^{m} q^{-(a+i)(b+i)}\times \tag{2}$$

$$\times\frac{(q)_{a+b}(q)_{2n+1}^{m+1}((q))_{2n-m}((q))_m((q))_a((q))_b((q))_{m+a+b}((q))_{2n-m-a-b}((q))_{n-m-a-1}((q))_{n-m-b-1}}{(q)_a(q)_b((q))_{2n+1}((q))_{a+b}((q))_{n-a}((q))_{n-b}((q))_{m+a}((q))_{m+b}((q))_{2n-2m-a-b-1}}.$$

Proof: The ratio of $R_m(a,b)$ in the recurrence implied by *Lewis* [AE] is:

$$\frac{R_{m-1}(a,b)R_{m-1}(a+1,b+1)}{R_m(a,b)R_{m-2}(a+1,b+1)}-\frac{R_{m-1}(a+1,b)R_{m-1}(a,b+1)}{R_m(a,b)R_{m-2}(a+1,b+1)}=$$

$$=\frac{(1-q^{2n-m-a-b+1})(1-q^{m+a+b+1})}{(1-q^{2n-m+2})(1-q^m)}-q^m\frac{(1-q^{2n-2m-a-b+1})(1-q^{a+b+1})}{(1-q^{2n-m+2})(1-q^m)},$$

which certainly reduces to 1. (scripts are found at `http://www.math.temple.edu/~tewodros`) □

References

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`The package EKHAD is available by the www at http://www.math.temple.edu/~zeilberg/programs.html`

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