A Q-GENERALIZATION OF A DETERMINANT EVALUATION CONJECTURED BY G. KUPERBERG AND J. PROPP

Tewodros Amdeberhan

The determinant evaluation:

$$det\left[\left(q^{\binom{i-j+1}{2}}\binom{i+j}{i}_q\binom{2n-i-j}{n-i}_q\right)_{0\le i,j\le n}\right]=\prod_{i=0}^{n}\binom{2n+1}{i}_q=\frac{(q)_{2n+1}^{n+1}}{((q))_{2n+1}} \tag{1}$$

was conjectured by G. Kuperberg and J. Propp [P], when q=1.

Let $M_n^q := [(q^{\binom{i-j+1}{2}}\binom{i+j}{i}_q\binom{2n-i-j}{n-i}_q)_{0\le i,j\le n}]$ and define the matrices:

$$M_{n,m}^q(a,b):=\left[\left(q^{\binom{i-j+a-b+1}{2}}\binom{i+j+a+b}{i+a}_q\binom{2n-i-j-a-b}{n-i-a}_q\right)_{0\le i,j\le m}\right].$$

PROPPosition: For the submatrices of M_n^q it holds that,

$$det M_{n,m}^q(a,b)=q^{(m+1)\binom{a-b+1}{2}}\times \tag{2}$$

$$\times\frac{(q)_{a+b}(q)_{2n+1}^{m+1}((q))_{2n-m}((q))_m((q))_a((q))_b((q))_{m+a+b}((q))_{2n-m-a-b}((q))_{n-m-a-1}((q))_{n-m-b-1}}{(q)_a(q)_b((q))_{2n+1}((q))_{a+b}((q))_{n-a}((q))_{n-b}((q))_{m+a}((q))_{m+b}((q))_{2n-2m-a-b-1}}.$$

Proof: The ratio of $R_m(a,b)$ in the recurrence implied by *Lewis* [AE] is:

$$\frac{R_{m-1}(a,b)R_{m-1}(a+1,b+1)}{R_m(a,b)R_{m-2}(a+1,b+1)}-\frac{R_{m-1}(a+1,b)R_{m-1}(a,b+1)}{R_m(a,b)R_{m-2}(a+1,b+1)}=$$

$$=\frac{(1-q^{2n-m-a-b+1})(1-q^{m+a+b+1})}{(1-q^{2n-m+2})(1-q^m)}-q^m\frac{(1-q^{2n-2m-a-b+1})(1-q^{a+b+1})}{(1-q^{2n-m+2})(1-q^m)},$$

which certainly reduces to 1. (scripts are found at `http://www.math.temple.edu/~tewodros`) □

References

[AE] T. Amdeberhan, S.B. Ekhad, *A condensed condensation proof of a determinant evaluation conjectured by Greg Kuperberg and Jim Propp,*, Jour. Comb. Theory (A).

[P] J. Propp, *E-mail message to D. Zeilberger*, dated 1 July 1996, 15:47:56 (EDT).

[PWZ] M. Petkovšek, H.S. Wilf, D.Zeilberger, *"A=B"*, A.K. Peters Ltd., 1996.
`The package EKHAD is available by the www at http://www.math.temple.edu/~zeilberg/programs.html`

[Z] D. Zeilberger, *Reverend Charles to the aid of Major Percy and Fields Medalist Enrico*, Amer. Math. Monthly **103** (1996), 501-502.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX