

ROCKY MOUNTAIN JOURNAL OF MATHEMATICS Volume 54 (2024), No. 6, 1527–1534

GAUSSIAN INEQUALITY

TEWODROS AMDEBERHAN AND DAVID CALLAN

We prove some special cases of Bergeron's inequality involving two Gaussian polynomials (or q-binomials).

1. Introduction

We begin by recalling the q-analogues

$$[n]!_q = \prod_{j=1}^n \frac{1-q^j}{1-q}$$

of factorials and the q-analogue

$$\binom{n}{k}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}$$

of binomial coefficients. Adopt the convention $[0]!_q = 1$. It is well-known that these rational functions $\binom{n}{k}_q$ are polynomials, in q, also called *Gaussian polynomials*, having nonnegative coefficients which are also *unimodal* and symmetric. Furthermore, there are several combinatorial interpretations of which we state two of them.

A word of length n over the alphabet set $\{0, 1\}$ is a finite sequence $w = a_1 \cdots a_n$. Construct

 $\mathcal{W}_{n,k} = \{w = a_1 \cdots a_n : w \text{ has } k \text{ zeros and } n - k \text{ ones}\}$

and the *inversion set* of w as $Inv(w) = \{(i, j) : i < j \text{ and } a_i > a_j\}$. The corresponding *inversion number* of w will be denoted inv(w) = #Inv(w). Then, we have

$$\binom{n}{k}_q = \sum_{w \in \mathcal{W}_{n,k}} q^{\operatorname{inv}(w)}.$$

Yet, another formulation which would come to appeal to many combinatorialists is

$$\binom{a+d}{a}_q = \sum_T q^{\operatorname{area}(T)}$$

where T is a lattice path inside an $a \times d$ box and area(T) is area above the curve T.

Given two polynomials f(q) and g(q), we write $f(q) \ge g(q)$ provided that f(q) - g(q) has nonnegative coefficients in the powers of q.

DOI: 10.1216/rmj.2024.54.1527

²⁰²⁰ AMS *Mathematics subject classification:* primary 11B65; secondary 11A07.

Keywords and phrases: positivity, Gaussian polynomials, Bergeron's inequality. Received by the editors on April 6, 2023.

The well-known *Foulkes conjecture* (see, for instance [4]) was generalized by Vessenes [8]. She conjectured that

(1)
$$(h_b \circ h_c) - (h_a \circ h_d)$$

is *Schur positive* (expands with positive integer coefficients in the Schur basis $\{s_{\mu}\}_{\mu \vdash n}$ of symmetric polynomials) whenever $a \leq b < c \leq d$, with n = ad = bc, and one writes $(h_n \circ h_k)$ for the *plethysm* of complete homogeneous symmetric functions. A well-known fact is that $(h_n \circ h_k)(1, q) = {\binom{n+k}{k}}_q$. Moreover, any nonzero evaluation of a Schur function at 1 and q is of the form $q^i + q^{i+1} + \cdots + q^j$ for some i < j. Exploiting these facts on the occasion of [3], and assuming that Schur positivity of (1) holds, F. Bergeron (see [4]) underlined that the evaluation of the difference in (1), at 1 and q, would imply the following:

Conjecture 1. Assume $0 < a \le b < c \le d$ are positive integers with ad = bc. Then, the following difference of two Gaussian polynomials is symmetric:

(2)
$$\binom{b+c}{b}_q - \binom{a+d}{a}_q \ge 0$$

One can associate a direct combinatorial meaning to Vessenes' conjecture in the context of representation theory of GL(V). Indeed, if it holds true, it would signify that there is an embedding of the composite of symmetric powers $S^a(S^d(V))$ inside $S^b(S^c(V))$, as GL(V)-modules. It may however be more natural to state that there is a surjective GL(V)-module morphism the other way around (which is also equivalent). Therefore each GL(V)-irreducible occurs with smaller multiplicity in $S^a(S^d(V))$ than it does in $S^b(S^c(V))$, and the conjecture reflects this at the level of the corresponding characters (with Schur polynomials appearing as characters of irreducible representations).

The sole attempt [9] toward resolving Conjecture 1 was made by F. Zanello, who attends to the special case $a \le 3$, including the property of symmetry and unimodality. A sequence of numbers is *unimodal* if it does not increase strictly after a strict decrease. Zanello [9] offered a strengthening of Conjecture 1 to the effect that:

Conjecture 2. Preserve the hypothesis in Conjecture 1. Then, the coefficients of the symmetric polynomial

$$\binom{b+c}{b}_q - \binom{a+d}{a}_q$$

are nonnegative and unimodal.

Notice that symmetry is clear, since both $\binom{b+c}{b}_q$ and $\binom{a+d}{a}_q$ are symmetric polynomials of the same degree, ad = bc. We started out this project with the goal of proving the below 3-parameter special case of Conjecture 1 which we dubbed the β -conjecture. Namely:

Conjecture 3. For integers 0 < a < b and $\beta \ge 1$, we have

(3)
$$\binom{b+\beta a}{b}_q \ge \binom{a+\beta b}{a}_q.$$

The case $\beta = 1$ is trivial. However, our journey in this effort failed short of capturing the β -conjecture in its fullest. In the sequel, we supply the details of our success in settling the particular instance $\beta = 2$. Let's commence by stating one useful identity.

Theorem 4 (q-analogue Vandermonde–Chu). The following holds true:

(4)
$$\sum_{j\geq 0} {\binom{X}{Z-j}}_q {\binom{Y}{j}}_q q^{j(X-Z+j)} = {\binom{X+Y}{Z}}_q.$$

Remark 5. In view of Theorem 4, Conjecture 1 is tantamount to

$$\binom{b+c}{b}_q = \sum_{k=0}^{b} \binom{b}{k}_q \binom{c}{k}_q q^{k^2} \ge \sum_{k=0}^{a} \binom{a}{k}_q \binom{d}{k}_q q^{k^2} = \binom{a+d}{a}_q$$

2. The case $\beta = 2$ and q = 1

In this section, we explain the resolution of the Conjecture 3 (the β -conjecture) for the ordinary binomial coefficients (q = 1) while $\beta = 2$, which elaborates a natural course of our work development.

For a < b < c < d with ad = bc and special case c = 2a, d = 2b, say b = a + i, $i \ge 1$, it would be desirable to find a bijective proof for

$$\binom{3a+i}{a+i} \ge \binom{3a+2i}{a}.$$

An injection from a set counted by the smaller number to one counted by the larger number would be nice but a better proof would be an expression for the difference as a sum of obviously positive terms. For i = 1, we have

$$\binom{3a+1}{a+1} - \binom{3a+2}{a} = \binom{3a+1}{a-1},$$

and the right-hand side is clearly positive. It seems for general i = 1, 2, ... that

$$\binom{3a+i}{a+i} - \binom{3a+2i}{a} = \sum_{k=1}^{i} c_k(i) \binom{3a+i}{a-k}$$

for integers $c_k(i)$ and, furthermore, the $c_k(i)$ are all positive. Here is a table for $c_k(i)$ when $1 \le k \le i \le 8$:

but it is hard to get a handle on them. The evolution of our next progress begins with the discovery of

$$c_k(i) = \binom{i+k-1}{2k} + 2\binom{i+k-1}{2k-1} - \binom{i}{k}.$$

Let's contract these coefficients as

$$c_k(i) = \frac{i+3k}{i+k} \binom{i+k}{2k} - \binom{i}{k},$$

for $i, k \ge 1$. Notice $c_0(i) = 0$. We need some preliminary results.

Lemma 6. We have

$$\binom{3a+2i}{a} = \sum_{k\geq 0} \binom{i}{k} \binom{3a+i}{a-k}.$$

Proof. This follows from the Vandermonde–Chu identity (Theorem 4 for q = 1)

$$\binom{X+Y}{Z} = \sum_{k\geq 0} \binom{X}{k} \binom{Y}{Z-k}$$

applied to $\binom{3a+2i}{a} = \binom{i+3a+i}{a}$ with X = i, Y = 3a + i and Z = a.

Lemma 7. We have

$$\binom{3a+i}{a+i} = \sum_{k\geq 0} \frac{i+3k}{i+k} \binom{i+k}{2k} \binom{3a+i}{a-k} = \sum_{k\geq 0} \left[\binom{i+k}{2k} + \binom{i+k-1}{2k-1} \right] \binom{3a+i}{a-k}$$

Proof. We implement Zeilberger's algorithm (from Wilf-Zeilberger theory). Define

$$F(i,k) = \frac{i+3k}{i+k} \cdot \frac{\binom{i+k}{2k}\binom{3a+i}{a-k}}{\binom{3a+i}{a+i}} \quad \text{and} \quad G(i,k) = -\frac{\binom{i+k-1}{2k-2}\binom{3a+i}{a-k}}{\binom{3a+i}{a+i}}$$

Check that F(i + 1, k) - F(i, k) = G(i, k + 1) - G(i, k) and sum both sides over all integer values k. Then, notice the right-hand side vanishes and hence we obtain a sum $\sum_{k} F(i, k)$ that is *constant* in the variable *i*. Determine this constant by substituting, say i = 1,

$$\sum_{k=0}^{1} F(1,k) = \frac{\binom{3a+1}{a}}{\binom{3a+1}{a+1}} + \frac{2\binom{3a+1}{a-1}}{\binom{3a+1}{a+1}} = \frac{a+1}{2a+1} + \frac{a}{2a+1} = 1.$$

Therefore, $\sum_{k} F(i, k) = 1$, identically, for all $i \ge 1$. The proof follows.

We now state the main result of this section.

Theorem 8. We have

$$\binom{3a+i}{a+i} - \binom{3a+2i}{a} = \sum_{k\geq 1} \left\{ \frac{i+3k}{i+k} \binom{i+k}{2k} - \binom{i}{k} \right\} \binom{3a+i}{a-k}$$

Proof. This is immediate from Lemmas 6 and 7.

Lemma 9. For $k \ge 1$, the coefficients $c_k(i)$ are nonnegative.

Proof. We may look at it in two different ways:

(1)
$$c_k(i) = \frac{2k}{i+k} \binom{i+k}{2k} + \binom{i+k}{2k} - \binom{i}{k} = \frac{2k}{i+k} \binom{i+k}{2k} + \binom{i+k}{i-k} - \binom{i}{i-k}$$
. Obviously $\binom{i+k}{i-k} \ge \binom{i}{i-k}$, therefore $c_k(i) \ge 0$.
(2) $c_k(i) = \binom{i+k}{2k} + \binom{i+k-1}{2k-1} - \binom{i}{k} = \binom{i+k}{2k} + \sum_{r=0}^{k-2} \binom{i+r}{k+1+r}$ shows clearly that $c_k(i) \ge 0$. The identity

$$\binom{i+k-1}{2k-1} = \binom{i}{k} + \sum_{r=1}^{k-1} \binom{i+r-1}{k+r}$$

results from a cascading effect of the familiar binomial recurrence $\binom{u}{v} + \binom{u}{v-1} = \binom{u+1}{v}$.

1530

3. *q*-analogues when $\beta = 2$

In the present section, we aim to generalize our proofs given in the preceding section by lifting the argument from the ordinary binomials to Gaussian polynomials.

Lemma 10. We have

$$\binom{3a+2i}{a}_q = \sum_{k\geq 0} q^{(a-k)(i-k)} \binom{i}{k}_q \binom{3a+i}{a-k}_q.$$

Proof. This follows from the q-Vandermonde-Chu identity (Theorem 4)

$$\binom{X+Y}{Z}_q = \sum_{k\geq 0} q^{(Z-k)(X-k)} \binom{X}{k}_q \binom{Y}{Z-k}_q$$

on $\binom{3a+2i}{a}_q = \binom{i+3a+i}{a}_q$ with X = i, Y = 3a+i and Z = a.

Lemma 11. We have

$$\binom{3a+i}{a+i}_q = \sum_{k\geq 0} q^{(a-k)(i-k)} \left[\binom{i+k}{2k}_q + q^{a+i} \binom{i+k-1}{2k-1}_q \right] \binom{3a+i}{a-k}_q.$$

Proof. Let's rewrite

$$\binom{i+k}{2k}_{q} + q^{a+i} \binom{i+k-1}{2k-1}_{q} = \left[1 + \frac{q^{a+i}(1-q^{2k})}{1-q^{i+k}}\right] \binom{i+k}{2k}_{q}$$

and define the functions

$$\begin{split} F(i,k) &= q^{(a-k)(i-k)} \bigg[1 + q^{a+i} \cdot \frac{1 - q^{2k}}{1 - q^{i+k}} \bigg] \frac{\binom{i+k}{2k}_q \binom{3a+i}{a-k}_q}{\binom{3a+i}{a+i}_q}, \\ G(i,k) &= -q^{(a-k+1)(i-k+1)} \cdot \frac{\binom{i+k-1}{2k-2}_q \binom{3a+i}{a-k}_q}{\binom{3a+i}{a+i}_q}. \end{split}$$

Divide both sides of the intended identity by $\binom{3a+i}{a+i}_q$. Our goal is to prove $\sum_k F(i, k) = 1$ by adopting the Wilf–Zeilberger technique. To this end, calculate the two ratios

$$A(i,k) := \frac{F(i+1,k)}{F(i,k)} - 1 \quad \text{and} \quad B(i,k) := \frac{G(i,k+1)}{F(i,k)} - \frac{G(i,k)}{F(i,k)}$$

resulting in

$$\begin{split} A(i,k) &= \frac{q^{a-k}(1-q^{i+k})(1-q^{a+i+1})(1-q^{i+k+1}+q^{a+i+1}-q^{a+i+2k+1})}{(1-q^{i-k+1})(1-q^{2a+i+k+1})(1-q^{i+k}+q^{a+i}-q^{a+i+2k})} - 1\\ B(i,k) &= \left[-\frac{1-q^{a-k}}{1-q^{2a+i+k+1}} + \frac{q^{a+i-2k+1}(1-q^{2k})(1-q^{2k-1})}{(1-q^{i+k})(1-q^{i-k+1})} \right] \cdot \frac{1-q^{i+k}}{1-q^{i+k}+q^{a+i}-q^{a+i+2k}}. \end{split}$$

Verify routinely A(i, k) = B(i, k). Thus F(i + 1, k) - F(i, k) = G(i, k + 1) - G(i, k). Now, sum both sides over all integer values k. Then, notice that the right-hand side vanishes and hence we obtain a sum $\sum_{k} F(i, k)$ that is *constant* in the variable i. Determine this constant by substituting, say i = 1 and proceed with some simplifications leading to

$$\sum_{k=0}^{1} F(1,k) = q^{a} \cdot \frac{1-q^{a+1}}{1-q^{2a+1}} + \frac{1-q^{a}}{1-q^{2a+1}} = 1.$$

Therefore, $\sum_{k} F(i, k) = 1$, identically, for all $i \ge 1$. The assertion follows.

Lemma 12. We have the identity

$$\binom{i+k}{2k}_q = \binom{i}{k}_q + \sum_{r=1}^k q^{k+r} \binom{i+r-1}{k+r}_q.$$

Proof. Use the recursive relations $\binom{a}{b}_q = \binom{a-1}{b}_q + q^{a-b}\binom{a-1}{b-1}_q = q^b\binom{a-1}{b}_q + \binom{a-1}{b-1}_q$.

Lemma 13. We have the inequality $\binom{i+k}{i-k}_q \ge \binom{i}{i-k}_q$.

Proof. We use the interpretation of the Gaussian polynomials as the *inversion number* generating function for all bit strings of length n with k zeroes and n - k ones, that is

$$\binom{n}{k}_q = \sum_{w \in 0^k 1^{n-k}} q^{\operatorname{inv}(w)}.$$

Let $w' \in 0^{i-k} 1^k \sqcup 1^k$ denote a bit where the last k digits are all ones. In this sense, we get

$$\binom{i+k}{i-k}_{q} = \sum_{w \in 0^{i-k}1^{2k}} q^{\operatorname{inv}(w)} = \sum_{w' \in 0^{i-k}1^{k} \sqcup 1^{k}} q^{\operatorname{inv}(w')} + \sum_{w' \notin 0^{i-k}1^{k} \sqcup 1^{k}} q^{\operatorname{inv}(w')}$$
$$= \sum_{w \in 0^{i-k}1^{k}} q^{\operatorname{inv}(w)} + \sum_{w' \notin 0^{i-k}1^{k} \sqcup 1^{k}} q^{\operatorname{inv}(w')}$$
$$= \binom{i}{i-k}_{q} + \sum_{w' \notin 0^{i-k}1^{k} \sqcup 1^{k}} q^{\operatorname{inv}(w')},$$

where we note that inv(w') = inv(w) if the word $w' \in 0^{i-k}1^k \sqcup 1^k$ is associated with $w \in 0^{i-k}1^k$ found by dropping the last k ones. The assertion is now immediate.

We prove the main result of this section and our paper, the β -conjecture for $\beta = 2$.

Theorem 14. The polynomial $P(q) := {\binom{3a+i}{a+i}}_q - {\binom{3a+2i}{a}}_q$ has nonnegative coefficients.

Proof. From Lemmas 10 and 11, we infer

$$P(q) = \sum_{k \ge 1} q^{(a-k)(i-k)} \left[\binom{i+k}{2k}_q + q^{a+i} \binom{i+k-1}{2k-1} - \binom{i}{k}_q \right] \binom{3a+i}{a-k}_q.$$

It suffices to verify positivity of the terms inside the inner parenthesis on the right-hand side. We may pair up these terms and complement the lower index to the effect that

$$\binom{i+k}{2k}_{q} - \binom{i}{k}_{q} + q^{a+i}\binom{i+k-1}{2k-1}_{q} = \binom{i+k}{i-k}_{q} - \binom{i}{i-k}_{q} + q^{a+i}\binom{i+k-1}{2k-1}_{q}.$$

To reach the conclusion, simply apply Lemmas 12 or 13.

4. Final remarks

In the present section, we close our discussion with one conjecture as a codicil of certain calculations we encountered while digging up ways to prove the β -conjecture (Conjecture 3).

Conjecture 15. For each $0 \le k \le a < b$, we have

$$\binom{a}{k}_{q}\binom{a+b}{b-k}_{q} \ge \binom{b}{k}_{q}\binom{a+b}{a-k}_{q} \quad or \quad \binom{a}{k}_{q}\binom{b}{k}_{q}\binom{b+a}{b}_{q}\left[\frac{1}{\binom{a+k}{k}_{q}} - \frac{1}{\binom{b+k}{k}_{q}}\right] \ge 0.$$

The next elementary result might be helpful if one decides to engage this conjecture. Lemma 16. For $0 \le k \le a < b$, we have

$$\frac{1}{\binom{a+k}{k}_q} - \frac{1}{\binom{b+k}{k}_q} = \sum_{i=1}^k q^{a+i} \frac{1-q^{b-a}}{1-q^{b+i}} \prod_{j=i}^k \frac{1-q^j}{1-q^{a+j}} \prod_{j=1}^{i-1} \frac{1-q^j}{1-q^{b+j}}.$$

Proof. This results from partial fractions.

Example 17.

$$\frac{1}{\binom{a+1}{1}_q} - \frac{1}{\binom{b+1}{1}_q} = \frac{q^{a+1}(1-q)(1-q^{b-a})}{(1-q^{a+1})(1-q^{b+1})},$$

$$\frac{1}{\binom{a+2}{2}_q} - \frac{1}{\binom{b+2}{2}_q} = \frac{q^{a+1}(1-q)(1-q^2)(1-q^{b-a})}{(1-q^{a+1})(1-q^{a+2})(1-q^{b+1})} + \frac{q^{a+2}(1-q)(1-q^2)(1-q^{b-a})}{(1-q^{a+2})(1-q^{b+1})(1-q^{b+2})}.$$

Example 18.

$$\begin{split} \frac{1}{\binom{a+3}{3}_q} &- \frac{1}{\binom{b+3}{3}_q} = \frac{q^{a+1}(1-q)(1-q^2)(1-q^3)(1-q^{b-a})}{(1-q^{a+1})(1-q^{a+2})(1-q^{a+3})(1-q^{b+1})} \\ &+ \frac{q^{a+2}(1-q)(1-q^2)(1-q^3)(1-q^{b-a})}{(1-q^{a+2})(1-q^{a+3})(1-q^{b+1})(1-q^{b+2})} \\ &+ \frac{q^{a+3}(1-q)(1-q^2)(1-q^3)(1-q^{b-a})}{(1-q^{a+3})(1-q^{b+1})(1-q^{b+2})(1-q^{b+3})}. \end{split}$$

Remark 19. As a side note, we recall that G. E. Andrews [2] expresses $\binom{n}{k}_{q} - \binom{n}{k-1}_{q}$ as the generating function for partitions with particular *Frobenius symbols*, while L. M. Butler [5] does this with the help of the *Kostka–Foulkes polynomials* to show nonnegativity of the coefficients. We shall provide an alternative algebraic approach.

1533

Lemma 20. For $0 \le 2k \le n$, we have $\binom{n}{k}_q - \binom{n}{k-1}_q \ge 0$. *Proof.* Let $n = \alpha k + d$, where $0 \le d \le k$. Rewrite

$$\binom{n}{k}_{q} - \binom{n}{k-1}_{q} = q^{k} \binom{n}{k-1}_{q} \frac{1 - q^{(\alpha-2)k}}{1 - q^{k}} + q^{(\alpha-1)k} \binom{n}{k-1}_{q} \frac{1 - q^{d+1}}{1 - q^{k}}$$

Observe $\frac{1-q^{(\alpha-2)k}}{1-q^k}$ is already a polynomial with nonnegative coefficients. Furthermore, since $U(q) := \binom{n}{k-1}_q$ is unimodal [1; 6; 7], the coefficient of q^j in $U(q) \cdot (1-q^{d+1})$ is nonnegative as long as $2j \le \deg(U)$. The same is true for $U(q)\frac{1-q^{d+1}}{1-q^k}$ as a formal power series. Since the *polynomial* $U(q)\frac{1-q^{d+1}}{1-q^k}$ is *symmetric*, having degree no greater than U(q), all remaining coefficients of $U(q)\frac{1-q^{d+1}}{1-q^k}$ are nonnegative. \Box

Acknowledgement

Amdeberhan thanks R. P. Stanley for bringing Bergeron's conjecture to his attention, also F. Bergeron for his generous explanation of the problem studied in the present work.

References

- G. E. Andrews, "A theorem on reciprocal polynomials with applications to permutations and compositions", *Amer. Math. Monthly* 82:8 (1975), 830–833.
- [2] G. E. Andrews, "On the difference of successive Gaussian polynomials", J. Statist. Plann. Inference 34:1 (1993), 19–22.
- [3] F. Bergeron, "The q-Foulkes' conjecture", talk, Bowdoin College, ME, 2016.
- [4] F. Bergeron, "A q-analog of Foulkes' conjecture", Electron. J. Combin. 24:1 (2017), art. id. 1.38.
- [5] L. M. Butler, "A unimodality result in the enumeration of subgroups of a finite abelian group", *Proc. Amer. Math. Soc.* 101:4 (1987), 771–775.
- [6] J. W. B. Hughes and J. Van der Jeugt, "Unimodal polynomials associated with Lie algebras and superalgebras", J. Comput. Appl. Math. 37:1-3 (1991), 81–88.
- [7] K. B. Stolarsky, *Higher partition functions and their relation to finitely generated nilpotent groups*, Ph.D. thesis, University of Wisconsin, 1968.
- [8] R. Vessenes, "Generalized Foulkes' conjecture and tableaux construction", J. Algebra 277:2 (2004), 579-614.
- [9] F. Zanello, "On Bergeron's positivity problem for q-binomial coefficients", Electron. J. Combin. 25:2 (2018), art. id. 2.17.

TEWODROS AMDEBERHAN: tamdeber@tulane.edu Department of Mathematics, Tulane University, New Orleans, LA, United States

DAVID CALLAN: callan@stat.wisc.edu Department of Statistics, University of Wisconsin-Madison, Madison, WI, United States