GAUSSIAN INEQUALITY

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Abstract. We prove some special cases of Bergeron's inequality involving two Gaussian polynomials (or q-binomials).

1. Introduction

We begin by recalling the q-analogues $[n]!_q = \prod_{j=1}^{n} \frac{1-q^j}{1-q}$ of factorials and the q-analogue $\binom{n}{k}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}$ of binomial coefficients. Adopt the convention $[0]!_q = 1$. It is well-known that these rational functions $\binom{n}{k}_q$ are polynomials, in q, also called *Gaussian polynomials*, having non-negative coefficients which are also *unimodal* and symmetric. Furthermore, there are several combinatorial interpretations of which we state two of them.

A *word* of length n over the *alphabet* set $\{0,1\}$ is a finite sequence $w = a_1 \cdots a_n$. Construct

$$\mathcal{W}_{n,k} = \{w = a_1 \cdots a_n : w \text{ has } k \text{ zeros and } n-k \text{ ones}\}$$

and the *inversion set* of w as $\mathrm{Inv}(w) = \{(i,j) : i < j \text{ and } a_i > a_j\}$. The corresponding *inversion number* of w will be denoted $\mathrm{inv}(w) = \#\mathrm{Inv}(w)$. Then, we have

$$\binom{n}{k}_q = \sum_{w \in \mathcal{W}_{n,k}} q^{\mathrm{inv}(w)}.$$

Yet, another formulation which would come to appeal to many combinatorialists is

$$\binom{a+d}{a}_q = \sum_T q^{area(T)}$$

where T is a lattice path inside an $a \times d$ box and $area(T)$ is area above the curve T.

Given two polynomials $f(q)$ and $g(q)$, we write $f(q) \geq g(q)$ provided that $f(q) - g(q)$ has non-negative coefficients in the powers of q.

The well-known *Foulkes conjecture* (see, for instance [4]) was generalized by Vessenes [8]. She conjectured that

$$(h_b \circ h_c) - (h_a \circ h_d) \tag{1}$$

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is *Schur positive* (expands with positive integer coefficients in the Schur basis $\{s_\mu\}_{\mu\vdash n}$ of symmetric polynomials) whenever $a \le b < c \le d$, with $n = ad = bc$, and one writes $(h_n \circ h_k)$ for the *plethysm* of complete homogeneous symmetric functions. A well-known fact is that $(h_n \circ h_k)(1, q) = \binom{n+k}{k}_q$. Moreover, any non-zero evaluation of a Schur function at 1 and q is of the form $q^i + q^{i+1} + \cdots + q^j$ for some $i < j$. Exploiting these facts on the occasion of [3], and assuming that (1) holds, F. Bergeron (see also [4]) underlined that the evaluation of the difference in (1), at 1 and q, would imply the following:

Conjecture 1. *Assume $0 < a \le b < c \le d$ are positive integers with $ad = bc$. Then, the following difference of two Gaussian polynomials is symmetric and satisfies*

$$\binom{b+c}{b}_q - \binom{a+d}{a}_q \ge 0. \tag{2}$$

One can associate a direct combinatorial meaning to Vessenes' conjecture in the context of representation theory of $GL(V)$. Indeed, if it holds true, it would signify that there is an embedding of the composite of symmetric powers $S^a(S^d(V))$ inside $S^b(S^c(V))$, as $GL(V)$-modules. It may however be more natural to state that there is a surjective $GL(V)$-module morphism the other way around (which is also equivalent). Therefore each $GL(V)$-irreducible occurs with smaller multiplicity in $S^a(S^d(V))$ than it does in $S^b(S^c(V))$, and the conjecture reflects this at the level of the corresponding characters (with Schur polynomials appearing as characters of irreducible representations).

The sole attempt [9] toward resolving Conjecture 1 was made by F. Zanello, who attends to the special case $a \le 3$, including the property of symmetry and unimodality. A sequence of numbers is *unimodal* if it does not increase strictly after a strict decrease. The author in [9] offers a strengthening of Conjecture 1 to the effect that

Conjecture 2. *Preserve the hypothesis in Conjecture 1. Then, the coefficients of the symmetric polynomial*

$$\binom{b+c}{b}_q - \binom{a+d}{a}_q$$

are non-negative and unimodal.

Notice that symmetry is clear, since both $\binom{b+c}{b}_q$ and $\binom{a+d}{a}_q$ are symmetric polynomials of the same degree, $ad = bc$. We started out this project with the goal of proving the below 3-parameter special case of Conjecture 1 which we dubbed the *β-Conjecture*. Namely,

Conjecture 3. *For integers $0 < a < b$ and $\beta \ge 1$, we have*

$$\binom{b+\beta a}{b}_q \ge \binom{a+\beta b}{a}_q. \tag{3}$$

The case $\beta = 1$ is trivial. However, our journey in this effort failed short of capturing the β-Conjecture in its fullest. In the sequel, we supply the details of our success in settling the particular instance $\beta = 2$. Let's commence by stating one useful identity.

Theorem 1. *(q-analogue Vandermonde-Chu). The following holds true*

$$\sum_{j\geq 0}\binom{X}{Z-j}_q\binom{Y}{j}_q q^{j(X-Z+j)}=\binom{X+Y}{Z}_q. \tag{4}$$

Remark 2. In view of (4), Conjecture 1 tantamount

$$\binom{b+c}{b}_q=\sum_{k=0}^{b}\binom{b}{k}_q\binom{c}{k}_q q^{k^2}\geq\sum_{k=0}^{a}\binom{a}{k}_q\binom{d}{k}_q q^{k^2}=\binom{a+d}{a}_q.$$

2. THE CASE $\beta=2$ AND $q=1$

In this section, we wish to explain the resolution of the β-Conjecture for the ordinary binomial coefficients ($q=1$) while $\beta=2$, which elaborates a natural development.

For $a<b<c<d$ with $ad=bc$ and special case $c=2a, d=2b$, say $b=a+i$, $i\geq 1$, it would be desirable to find a bijective proof for

$$\binom{3a+i}{a+i}\geq\binom{3a+2i}{a}.$$

An injection from a set counted by the smaller number to one counted by the larger number would be nice but a better proof would be an expression for the difference as a sum of obviously positive terms. For $i=1$, we have

$$\binom{3a+1}{a+1}-\binom{3a+2}{a}=\binom{3a+1}{a-1},$$

and the right-hand side is clearly positive. It seems for general $i=1,2,\dots,$

$$\binom{3a+i}{a+i}-\binom{3a+2i}{a}=\sum_{k=1}^{i}c_k(i)\binom{3a+i}{a-k}$$

for integers $c_k(i)$ and, furthermore, the $c_k(i)$ *are all positive.* Here is a table for $c_k(i)$ when $1\leq k\leq i\leq 8$:

1							
3	1						
6	6	1					
10	19	9	1				
15	45	39	12	1			
21	90	120	66	15	1		
28	161	301	250	100	18	1	
36	266	658	755	450	141	21	1

but appeared hard to get a handle on them. The evolution of our next progress begins with the discovery of

$$c_k(i)=\binom{i+k-1}{2k}+2\binom{i+k-1}{2k-1}-\binom{i}{k}.$$

Let's contract these coefficients as $c_k(i) = \frac{i+3k}{i+k}\binom{i+k}{2k} - \binom{i}{k}$, for $i, k \geq 1$. Notice $c_0(i) = 0$. We need some preliminary results.

Lemma 1. *We have*

$$\binom{3a+2i}{a} = \sum_{k\geq 0}\binom{i}{k}\binom{3a+i}{a-k}.$$

Proof. This follows from the Vandermonde-Chu identity (Theorem 1 for $q = 1$)

$$\binom{X+Y}{Z} = \sum_{k\geq 0}\binom{X}{k}\binom{Y}{Z-k}$$

applied to $\binom{3a+2i}{a} = \binom{i+3a+i}{a}$ with $X = i, Y = 3a + i$ and $Z = a$. □

Lemma 2. *We have*

$$\binom{3a+i}{a+i} = \sum_{k\geq 0}\frac{i+3k}{i+k}\binom{i+k}{2k}\binom{3a+i}{a-k} = \sum_{k\geq 0}\left[\binom{i+k}{2k} + \binom{i+k-1}{2k-1}\right]\binom{3a+i}{a-k}.$$

Proof. We implement Zeilberger's algorithm (from the Wilf-Zeilberger theory). Define

$$F(i,k) = \frac{i+3k}{i+k}\cdot\frac{\binom{i+k}{2k}\binom{3a+i}{a-k}}{\binom{3a+i}{a+i}} \qquad \text{and} \qquad G(i,k) = -\frac{\binom{i+k-1}{2k-2}\binom{3a+i}{a-k}}{\binom{3a+i}{a+i}}.$$

Check that $F(i+1,k) - F(i,k) = G(i,k+1) - G(i,k)$ and sum both sides over all integer values k. Then, notice the right-hand side vanishes and hence we obtain a sum $\sum_k F(i,k)$ that is *constant* in the variable i. Determine this constant by substituting, say $i = 1$,

$$\sum_{k=0}^{1} F(1,k) = \frac{\binom{3a+1}{a}}{\binom{3a+1}{a+1}} + \frac{2\binom{3a+1}{a-1}}{\binom{3a+1}{a+1}} = \frac{a+1}{2a+1} + \frac{a}{2a+1} = 1.$$

Therefore, $\sum_k F(i,k) = 1$, identically, for all $i \geq 1$. The proof follows. □

We now state the main result of this section.

Theorem 3. *We have*

$$\binom{3a+i}{a+i} - \binom{3a+2i}{a} = \sum_{k\geq 1}\left\{\frac{i+3k}{i+k}\binom{i+k}{2k} - \binom{i}{k}\right\}\binom{3a+i}{a-k}.$$

Proof. Immediate from Lemma 1 and Lemma 2. □

Lemma 3. *For $k \geq 1$, the coefficients $c_k(i)$ are non-negative.*

Proof. We may look at it in two different ways:
(1) $c_k(i) = \frac{2k}{i+k}\binom{i+k}{2k} + \binom{i+k}{2k} - \binom{i}{k} = \frac{2k}{i+k}\binom{i+k}{2k} + \binom{i+k}{i-k} - \binom{i}{i-k}$. Obviously $\binom{i+k}{i-k} \geq \binom{i}{i-k}$, therefore $c_k(i) \geq 0$.
(2) $c_k(i) = \binom{i+k}{2k} + \binom{i+k-1}{2k-1} - \binom{i}{k} = \binom{i+k}{2k} + \sum_{r=0}^{k-2}\binom{i+r}{k+1+r}$ shows clearly that $c_k(i) \geq 0$. The identity $\binom{i+k-1}{2k-1} = \binom{i}{k} + \sum_{r=1}^{k-1}\binom{i+r-1}{k+r}$ results from a cascading effect of the familiar binomial recurrence $\binom{u}{v} + \binom{u}{v-1} = \binom{u+1}{v}$. □

3. q-ANALOGUES WHEN $\beta = 2$

In the present section, we aim to generalize our proofs given in the preceding section by lifting the argument from the ordinary binomials to Gaussian polynomials.

Lemma 4. *We have*

$$\binom{3a+2i}{a}_q = \sum_{k\geq 0} q^{(a-k)(i-k)} \binom{i}{k}_q \binom{3a+i}{a-k}_q.$$

Proof. This follows from the *Vandermonde-Chu* identity (Theorem 1)

$$\binom{X+Y}{Z}_q = \sum_{k\geq 0} q^{(Z-k)(X-k)} \binom{X}{k}_q \binom{Y}{Z-k}_q$$

on $\binom{3a+2i}{a}_q = \binom{i+3a+i}{a}_q$ with $X = i, Y = 3a+i$ and $Z = a$. □

Lemma 5. *We have*

$$\binom{3a+i}{a+i}_q = \sum_{k\geq 0} q^{(a-k)(i-k)} \left[\binom{i+k}{2k}_q + q^{a+i} \binom{i+k-1}{2k-1}_q \right] \binom{3a+i}{a-k}_q.$$

Proof. Let's rewrite $\binom{i+k}{2k}_q + q^{a+i}\binom{i+k-1}{2k-1}_q = \left[1 + \frac{q^{a+i}(1-q^{2k})}{1-q^{i+k}}\right]\binom{i+k}{2k}_q$ and define the functions

$$F(i,k) = q^{(a-k)(i-k)} \left[1 + q^{a+i} \cdot \frac{1-q^{2k}}{1-q^{i+k}}\right] \frac{\binom{i+k}{2k}_q \binom{3a+i}{a-k}_q}{\binom{3a+i}{a+i}_q}, \qquad \text{and}$$

$$G(i,k) = -q^{(a-k+1)(i-k+1)} \cdot \frac{\binom{i+k-1}{2k-2}_q \binom{3a+i}{a-k}_q}{\binom{3a+i}{a+i}_q}.$$

Divide both sides of the intended identity by $\binom{3a+i}{a+i}_q$. Our goal is to prove $\sum_k F(i,k) = 1$ by adopting the Wilf-Zeilberger technique. To this end, calculate the following two ratios

$$A(i,k) := \frac{F(i+1,k)}{F(i,k)} - 1 \qquad \text{and} \qquad B(i,k) := \frac{G(i,k+1)}{F(i,k)} - \frac{G(i,k)}{F(i,k)}$$

resulting in

$$A(i,k) = \frac{q^{a-k}(1-q^{i+k})(1-q^{a+i+1})(1-q^{i+k+1}+q^{a+i+1}-q^{a+i+2k+1})}{(1-q^{i-k+1})(1-q^{2a+i+k+1})(1-q^{i+k}+q^{a+i}-q^{a+i+2k})} - 1 \qquad \text{and}$$

$$B(i,k) = \left[-\frac{1-q^{a-k}}{1-q^{2a+i+k+1}} + \frac{q^{a+i-2k+1}(1-q^{2k})(1-q^{2k-1})}{(1-q^{i+k})(1-q^{i-k+1})}\right] \cdot \frac{1-q^{i+k}}{1-q^{i+k}+q^{a+i}-q^{a+i+2k}}.$$

Verify routinely $A(i,k) = B(i,k)$. Thus $F(i+1,k) - F(i,k) = G(i,k+1) - G(i,k)$. Now, sum both sides over all integer values k. Then, notice that the right-hand side vanishes and hence we obtain a sum $\sum_k F(i,k)$ that is *constant* in the variable i. Determine this constant by substituting, say $i=1$ and proceed with some simplifications leading to

$$\sum_{k=0}^{1} F(1,k) = q^a \cdot \frac{1-q^{a+1}}{1-q^{2a+1}} + \frac{1-q^a}{1-q^{2a+1}} = 1.$$

Therefore, $\sum_k F(i,k) = 1$, identically, for all $i \geq 1$. The assertion follows. $\square$

Lemma 6. *We have the identity*

$$\binom{i+k}{2k}_q = \binom{i}{k}_q + \sum_{r=1}^{k} q^{k+r} \binom{i+r-1}{k+r}_q.$$

Proof. Use the recursive relations $\binom{a}{b}_q = \binom{a-1}{b}_q + q^{a-b}\binom{a-1}{b-1}_q = q^b\binom{a-1}{b}_q + \binom{a-1}{b-1}_q$. $\square$

Lemma 7. *We have the inequality* $\binom{i+k}{i-k}_q \geq \binom{i}{i-k}_q$.

Proof. We use the interpretation of the Gaussian polynomials as the *inversion number* generating function for all bit strings of length n with k zeroes and $n-k$ ones, that is

$$\binom{n}{k}_q = \sum_{w \in 0^k 1^{n-k}} q^{\mathrm{inv}(w)}.$$

Let $w' \in 0^{i-k}1^k \sqcup 1^k$ denote a bit where the last k digits are all ones. In this sense, we get

$$\begin{aligned} \binom{i+k}{i-k}_q = \sum_{w \in 0^{i-k}1^{2k}} q^{\mathrm{inv}(w)} &= \sum_{w' \in 0^{i-k}1^k \sqcup 1^k} q^{\mathrm{inv}(w')} + \sum_{w' \notin 0^{i-k}1^k \sqcup 1^k} q^{\mathrm{inv}(w')} \\ &= \sum_{w \in 0^{i-k}1^k} q^{\mathrm{inv}(w)} + \sum_{w' \notin 0^{i-k}1^k \sqcup 1^k} q^{\mathrm{inv}(w')} \\ &= \binom{i}{i-k}_q + \sum_{w' \notin 0^{i-k}1^k \sqcup 1^k} q^{\mathrm{inv}(w')} \end{aligned}$$

where we note that $\mathrm{inv}(w') = \mathrm{inv}(w)$ if the word $w' \in 0^{i-k}1^k \sqcup 1^k$ is associated with $w \in 0^{i-k}1^k$ found by dropping the last k ones. The assertion is now immediate. $\square$

We prove the main result of this section and our paper, the β-Conjecture for $\beta = 2$.

Theorem 4. *The polynomial* $P(q) := \binom{3a+i}{a+i}_q - \binom{3a+2i}{a}_q$ *has non-negative coefficients.*

Proof. From Lemma 4 and Lemma 5, we infer

$$P(q) = \sum_{k\geq 1} q^{(a-k)(i-k)} \left[\binom{i+k}{2k}_q + q^{a+i}\binom{i+k-1}{2k-1} - \binom{i}{k}_q \right] \binom{3a+i}{a-k}_q.$$

It suffices to verify positivity of the terms inside the inner parenthesis on the right-hand side. We may pair up these terms and compliment the lower index to the effect that

$$\binom{i+k}{2k}_q-\binom{i}{k}_q+q^{a+i}\binom{i+k-1}{2k-1}_q=\binom{i+k}{i-k}_q-\binom{i}{i-k}_q+q^{a+i}\binom{i+k-1}{2k-1}_q.$$

To reach the conclusion, simply apply Lemma 6 or Lemma 7. □

4. Final remarks

In the present section, we close our discussion with one conjecture as a codicil of certain calculations we encountered while digging up ways to prove the β-Conjecture.

Conjecture 4. *For each* $0\leq k\leq a<b$, *we have*

$$\binom{a}{k}_q\binom{a+b}{b-k}_q\geq\binom{b}{k}_q\binom{a+b}{a-k}_q \qquad \textit{or} \qquad \binom{a}{k}_q\binom{b}{k}_q\binom{b+a}{b}_q\left[\frac{1}{\binom{a+k}{k}_q}-\frac{1}{\binom{b+k}{k}_q}\right]\geq 0.$$

The next elementary result might be helpful if one decides to engage this conjecture.

Lemma 8. *For* $0\leq k\leq a<b$, *we have*

$$\frac{1}{\binom{a+k}{k}_q}-\frac{1}{\binom{b+k}{k}_q}=\sum_{i=1}^{k}q^{a+i}\frac{1-q^{b-a}}{1-q^{b+i}}\prod_{j=i}^{k}\frac{1-q^j}{1-q^{a+j}}\prod_{j=1}^{i-1}\frac{1-q^j}{1-q^{b+j}}.$$

Proof. This results from partial fractions. □

Example 1.

$$\frac{1}{\binom{a+1}{1}_q}-\frac{1}{\binom{b+1}{1}_q}=\frac{q^{a+1}(1-q)(1-q^{b-a})}{(1-q^{a+1})(1-q^{b+1})}$$

$$\frac{1}{\binom{a+2}{2}_q}-\frac{1}{\binom{b+2}{2}_q}=\frac{q^{a+1}(1-q)(1-q^2)(1-q^{b-a})}{(1-q^{a+1})(1-q^{a+2})(1-q^{b+1})}+\frac{q^{a+2}(1-q)(1-q^2)(1-q^{b-a})}{(1-q^{a+2})(1-q^{b+1})(1-q^{b+2})}.$$

Example 2.

$$\begin{aligned}\frac{1}{\binom{a+3}{3}_q}-\frac{1}{\binom{b+3}{3}_q}&=\frac{q^{a+1}(1-q)(1-q^2)(1-q^3)(1-q^{b-a})}{(1-q^{a+1})(1-q^{a+2})(1-q^{a+3})(1-q^{b+1})}\\&+\frac{q^{a+2}(1-q)(1-q^2)(1-q^3)(1-q^{b-a})}{(1-q^{a+2})(1-q^{a+3})(1-q^{b+1})(1-q^{b+2})}\\&+\frac{q^{a+3}(1-q)(1-q^2)(1-q^3)(1-q^{b-a})}{(1-q^{a+3})(1-q^{b+1})(1-q^{b+2})(1-q^{b+3})}.\end{aligned}$$

Remark 5. As a side note, we recall that G. E. Andrews [2] expresses $\binom{n}{k}_q - \binom{n}{k-1}_q$ as the generating function for partitions with particular *Frobenius symbols*, while L. M. Butler [5] does this with the help of the *Kostka-Foulkes polynomials* to show non-negativity of the coefficients. We shall provide an alternative algebraic approach.

Lemma 9. *For* $0 \leq 2k \leq n$, *we have* $\binom{n}{k}_q - \binom{n}{k-1}_q \geq 0$.

Proof. Let $n = \alpha k + d$ where $0 \leq d < k$. Rewrite

$$\binom{n}{k}_q - \binom{n}{k-1}_q = q^k \binom{n}{k-1}_q \frac{1-q^{(\alpha-2)k}}{1-q^k} + q^{(\alpha-1)k} \binom{n}{k-1}_q \frac{1-q^{d+1}}{1-q^k}.$$

Observe $\frac{1-q^{(\alpha-2)k}}{1-q^k}$ is already a polynomial with non-negative coefficients. Furthermore, since $U(q) := \binom{n}{k-1}_q$ is unimodal [1], [6], [7], the coefficient of q^j in $U(q) \cdot (1-q^{d+1})$ is non-negative as long as $2j \leq \deg(U)$. The same is true for $U(q)\frac{1-q^{d+1}}{1-q^k}$ as a formal power series. Since the *polynomial* $U(q)\frac{1-q^{d+1}}{1-q^k}$ is *symmetric,* having degree no greater than $U(q)$, all remaining coefficients of $U(q)\frac{1-q^{d+1}}{1-q^k}$ are non-negative. ☐

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