SOLUTION TO PROBLEM #12432

Problem #12432. Proposed by Erik Vigren, Swedish Institute of Space Physics, Uppsala, Sweden. Suppose that k and n are integers with $n \geq 2$ and $1 \leq k \leq n$. What is the average value of $\sum_{i=1}^{\pi(k)} \pi(i)$ over all permutations π of $\{1, 2, \ldots, n\}$?

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, and Akalu Tefera, Grand Valley State University, MI, USA. Fix $1 \leq k \leq n$. Suppose $\pi(k) = j \in [n] := \{1, \ldots, n\}$.

Case 1: $j < k$**.** We're looking at the sums $\pi(1) + \cdots + \pi(j)$, each summand belonging to $[n] - \{j\}$, and appearing $\binom{n-2}{j-1}$ times. Furthermore, this j-element set can be permuted $j!$ times while the remaining $n - j - 1$ members form $(n - j - 1)!$ permutations. Thus, the total sum of all such permutations $\sum_{i=1}^{j} \pi(i)$ contributes the value

$$j!\,(n-j-1)!\binom{n-2}{j-1} \times \sum_{s\in[n]-\{j\}} s = j!\,(n-j-1)!\binom{n-2}{j-1}\left[\binom{n+1}{2} - j\right]. \tag{1}$$

Case 2: $j \geq k$**.** This time we are concerned with $\pi(1) + \cdots + \pi(k-1) + \boldsymbol{j} + \pi(k+1) + \cdots + \pi(j)$. The same argument as above reveals the value

$$(j-1)!\,(n-j)!\binom{n-2}{j-2}\left[\binom{n+1}{2} - j\right]. \tag{2}$$

In addition, we need to account for how often $\boldsymbol{j}$ appears as a summand. Direct enumeration gives

$$(j-1)!\,(n-j)!\binom{n-1}{j-1}\boldsymbol{j}. \tag{3}$$

Combining the results in (1), (2) and (3) shows an over all sum

$$\begin{aligned}\sum_{\pi\in\mathfrak{S}_n}\sum_{i=1}^{\pi(k)} \pi(i) &= \sum_{j=1}^{k-1} j!\,(n-j-1)!\binom{n-2}{j-1}\left[\binom{n+1}{2} - j\right] \\ &\quad + \sum_{j=k}^{n}(j-1)!\,(n-j)!\binom{n-2}{j-2}\left[\binom{n+1}{2} - j\right] + \sum_{j=k}^{n}\boldsymbol{j!}\,(n-j)!\binom{n-1}{j-1} \\ &= (n-2)!\sum_{j=1}^{k-1} j\left[\binom{n+1}{2} - j\right] + (n-2)!\sum_{j=k}^{n}(j-1)\left[\binom{n+1}{2} - j\right] + (n-1)!\sum_{j=k}^{n}\boldsymbol{j}.\end{aligned}$$

If we utilize $\sum_{a=1}^{N} a = \binom{N+1}{2}$ and $\sum_{a=1}^{N} a^2 = \frac{N(N+1)(2N+1)}{6}$, then routine simplification yields

$$\sum_{\pi\in\mathfrak{S}_n}\sum_{i=1}^{\pi(k)} \pi(i) = \frac{1}{2}(k-1)\,n\,(n-k+1)\,(n-2)! + \frac{1}{12}(3n+2)\,(n+1)!$$

and hence the desired average becomes $\frac{(k-1)(n-k+1)}{2(n-1)} + \frac{(n+1)(3n+2)}{12}$. $\square$

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