SOLUTION TO PROBLEM #12436

Problem #12436. Proposed by L. Sauras-Altuzarra (Vienna). For a positive integer n, evaluate

$$\prod_{k=1}^{n}\left(x+\sin^2\left(\frac{k\pi}{2n}\right)\right).$$

Solution by Tewodros Amdeberhan, Tulane University, New Orleans, LA, USA. We use the identity $2\sin^2\theta = 1-\cos(2\theta)$ and substitute $y = 2x+1$ to convert the given product to

$$\prod_{k=1}^{n}\left(x+\frac{1}{2}-\frac{1}{2}\cos\left(\frac{k\pi}{n}\right)\right)=\frac{1}{2^n}\prod_{k=1}^{n}\left(y-\cos\left(\frac{k\pi}{n}\right)\right)=\frac{y+1}{2^n}\prod_{k=1}^{n-1}\left(y-\cos\left(\frac{k\pi}{n}\right)\right).$$

However, the set $\{\cos(\frac{k\pi}{n})\}_{k=1}^{n-1}$ forms exactly the roots of the *Chebyshev polynomials of the* 2^{nd}*-kind, denoted* $U_{n-1}(y)$. Furthermore, we have that

$$\prod_{k=1}^{n-1}\left(y-\cos\left(\frac{k\pi}{n}\right)\right)=\sum_{j=0}^{\lfloor (n-1)/2\rfloor}\left(\frac{-1}{4}\right)^j\binom{n-j-1}{j}y^{n-1-2j}.$$

Combining all these results, we gather that

$$\prod_{k=1}^{n}\left(x+\sin^2\left(\frac{k\pi}{2n}\right)\right)=\frac{x+1}{2^{n-1}}\sum_{j=0}^{\lfloor \frac{n-1}{2}\rfloor}\left(\frac{-1}{4}\right)^j\binom{n-j-1}{j}(2x+1)^{n-1-2j}. \qquad \square$$