PROOF OF FORMULA 3.267.1

$$\int_0^1 \frac{x^{3n}\,dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}} \frac{\Gamma\left(n+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\,\Gamma(n+1)} \qquad n \in \mathbb{N}$$

Parameter restrictions. Convergence near $x=0$ requires $\operatorname{Re} n > -\frac{1}{3}$.

The stated formula is valid for (at least) $n \in \mathbb{R}$. It should be written as

$$\int_0^1 \frac{x^{3a}\,dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}} \frac{\Gamma\left(a+\frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)\,\Gamma(a+1)} \qquad a \in \mathbb{R}$$

In the special case $n \in \mathbb{N}$ it should be written as

$$\int_0^1 \frac{x^{3n}\,dx}{\sqrt[3]{1-x^3}} = \frac{2\pi}{3\sqrt{3}} \frac{\Gamma\left(n+\frac{1}{3}\right)}{n!\,\Gamma\left(\frac{1}{3}\right)} = \frac{2\pi}{3\sqrt{3}} \frac{\left(\frac{1}{3}\right)_n}{n!} \qquad n \in \mathbb{N}$$

Evaluation. The change of variables $t = x^3$ produces

$$\int_0^1 \frac{x^{3a}\,dx}{\sqrt[3]{1-x^3}} = \frac{1}{3}\int_0^1 \frac{t^{2a-3}\,dx}{(1-t)^{1/3}}.$$

The integral representation

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt \tag{1}$$

(which appears as entry **8.380.1**) gives the last integral as

$$B\left(a+\tfrac{1}{3}, \tfrac{2}{3}\right) = \frac{\Gamma\left(a+\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(a+1)}. \tag{2}$$

The form given in the table uses the relation

$$\Gamma(u)\Gamma(1-u) = \frac{\pi}{\sin \pi u} \tag{3}$$

(which appears as entry **8.334.3**) to obtain the first evaluation.