

PROOF OF FORMULA 3.411.18

$$\int_0^{\infty} \frac{x^3 e^{-nx} dx}{1 + e^{-x}} = 6 \sum_{k=n}^{\infty} \frac{(-1)^{n+k}}{k^4}$$

Write

$$\int_0^{\infty} \frac{x^3 e^{-nx} dx}{1 + e^{-x}} = \int_0^{\infty} \frac{x^3 e^{-(n-1)x} dx}{1 + e^x}.$$

Entry **3.411.8** states that

$$\int_0^{\infty} \frac{x^{n-1} e^{-px} dx}{1 + e^x} = (n-1)! \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k+p)^n}.$$

Replacing $n = 4$ and $p = n - 1$ gives

$$\int_0^{\infty} \frac{x^3 e^{-(n-1)x} dx}{1 + e^x} = 6 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k+n-1)^4}.$$

Now shift the index to obtain the result.