PROOF OF FORMULA 3.411.21

$$\int_0^\infty x^{n-1} \frac{1-e^{-mx}}{1-e^x} \, dx = -(n-1)! \sum_{k=1}^{m} \frac{1}{k^n}$$

Write the integral as

$$\int_0^\infty x^{n-1} \frac{1-e^{-mx}}{1-e^x} \, dx = -\int_0^\infty x^{n-1} e^{-mx} \frac{e^{mx}-1}{e^x-1} \, dx.$$

The rational function part of the integrand can be expanded to obtain

$$\int_0^\infty x^{n-1} \frac{1-e^{-mx}}{1-e^{-x}} \, dx = -\sum_{k=0}^{m-1} \int_0^\infty x^{n-1} e^{-(m-k)x} \, dx.$$

The change of variables $t = (m-k)x$ gives

$$\int_0^\infty x^{n-1} \frac{1-e^{-mx}}{1-e^{-x}} \, dx = -\sum_{k=0}^{m-1} \frac{1}{(m-k)^n} \int_0^\infty t^{n-1} e^{-t} \, dt.$$

The integral is recognized as $\Gamma(n) = (n-1)!$, to conclude.