PROOF OF FORMULA 3.429

$$\int_0^\infty \left[e^{-x} - (1+x)^{-a} \right] \frac{dx}{x} = \psi(a)$$

This is one of the fundamental integral representations of the digamma function. The proof presented here appears in *Special Functions* by Andrews, Askey and Roy.

Evaluate the double integral

$$\int_0^\infty \int_1^s e^{-xt} dt\, dx = \int_0^\infty \frac{e^{-x} - e^{-xs}}{x} ds.$$

Exchange the order of integration to obtain

$$\int_1^s \int_0^\infty e^{-xt} dt\, dx = \int_1^s \frac{dt}{t} = \ln s.$$

Therefore

$$\int_0^\infty \frac{e^{-x} - e^{-xs}}{x} dx = \ln s.$$

Now write

$$\int_0^\infty e^{-s} s^{a-1} \ln s\, ds = \int_0^\infty e^{-s} s^{a-1} \int_0^\infty \frac{e^{-x} - e^{-xs}}{x}\, dx\, ds.$$

The left hand side is $\Gamma'(a)$ and the right hand side is

$$\int_0^\infty \frac{1}{x} \left[e^{-x} \int_0^\infty s^{a-1} e^{-s} ds - \int_0^\infty s^{a-1} e^{-s(1+x)} ds \right] dx.$$

It follows that

$$\Gamma'(a) = \Gamma(a) \int_0^\infty \left[e^{-x} - (1+x)^{-a} \right] \frac{dx}{x}.$$