PROOF OF FORMULA 3.524.3

$$\begin{aligned}\int_0^\infty \frac{\sinh ax}{\sinh bx}\frac{dx}{x^p} &= \Gamma(1-p)\sum_{k=0}^\infty \left(\frac{1}{[b(2k+1)-a]^{1-p}} - \frac{1}{[b(2k+1)+a]^{1-p}} \right) \\ &= \frac{\Gamma(1-p)}{(2b)^{1-p}} \left[\zeta\left(\frac{b-a}{2b}, 1-p \right) - \zeta\left(\frac{b+a}{2b}, 1-p \right) \right]\end{aligned}$$

Write the integral as

$$\int_0^\infty \frac{\sinh ax}{\sinh bx}\frac{dx}{x^p} = \int_0^\infty \frac{e^{(a-b)x} - e^{-(a+b)x}}{1-e^{-2bx}} \frac{dx}{x^p}$$

and expanding the integrand as a geometric series we obtain

$$\int_0^\infty \frac{\sinh ax}{\sinh bx}\frac{dx}{x^p} = \sum_{k=0}^\infty \left[e^{-(2bk+b-a)x} - e^{-(2bk+b+a)x} \right] \frac{dx}{x^p}.$$

Entry **3.434.1** states that

$$\int_0^\infty \frac{e^{-\nu x} - e^{-\mu x}}{x^{r+1}}\, dx = \frac{\mu^r - \nu^r}{r}\Gamma(1-r).$$

The result now follows by choosing $r = p-1$. The representation of the series in terms of the Hurwitz zeta function is direct.