PROOF OF FORMULA 3.524.5

$$\int_0^\infty x^{s-1} \frac{\cosh bx}{\sinh cx} \, dx = \frac{\Gamma(s)}{(2c)^s} \left[\zeta\left(s, \tfrac{1}{2}\left(1 - \tfrac{b}{c}\right)\right) + \zeta\left(s, \tfrac{1}{2}\left(1 + \tfrac{b}{c}\right)\right)\right]$$

Write the integral as

$$\int_0^\infty x^{s-1} \frac{\cosh bx}{\sinh cx} \, dx = \int_0^\infty x^{s-1} (e^{bx} + e^{-bx}) e^{-cx} \frac{dx}{1 - e^{-2cx}}.$$

Now expand part of the integrand as a geometric series to produce

$$\int_0^\infty x^{s-1} \frac{\cosh bx}{\sinh cx} \, dx = \sum_{k=0}^\infty \int_0^\infty x^{s-1} e^{(b-c-2ck)x} \, dx + \sum_{k=0}^\infty \int_0^\infty x^{s-1} e^{(-b-c-2ck)x} \, dx.$$

Scaling the exponent yields

$$\int_0^\infty x^{s-1} \frac{\cosh bx}{\sinh cx} \, dx = \frac{\Gamma(s)}{(2c)^s} \left[\sum_{k=0}^\infty \frac{1}{(k + \frac{1}{2}(1 - b/c))^s} + \sum_{k=0}^\infty \frac{1}{(k + \frac{1}{2}(1 + b/c))^s} \right].$$

This is the stated expression, using the Hurwitz zeta function

$$\zeta(z, q) = \sum_{k=0}^\infty \frac{1}{(k+q)^z}.$$