PROOF OF FORMULA 3.527.2

$$\int_0^\infty \frac{x^{2m}\, dx}{\sinh^2(ax)} = \frac{\pi^{2m}}{a^{2m+1}} |B_{2m}|$$

The change of variable $t = ax$ shows that it is sufficient to consider the case $a = 1$. Start with

$$\int_0^\infty \frac{t^{\mu-1}\, dx}{\sinh^2 t} = 4 \int_0^\infty \frac{t^{\mu-1}\, dx}{(e^t - e^{-t})^2},$$

and write the last integral as

$$J := \int_0^\infty \frac{t^{\mu-1}\, dt}{(e^t - e^{-t})^2} = \int_0^\infty \frac{t^{\mu-1} e^{-2t}\, dt}{(1 - e^{-2t})^2}.$$

Expand in a power series to obtain

$$J = \sum_{n=1}^\infty n \int_0^\infty t^{\mu-1} e^{-2nt}\, dt.$$

The change of variable $v = 2nt$ yields

$$J = \sum_{n=1}^\infty \frac{1}{n^{\mu-1}} \times \frac{1}{2^\mu} \int_0^\infty v^{\mu-1} e^{-v}\, dv.$$

This shows

$$\int_0^\infty \frac{x^{\mu-1}\, dx}{\sinh^2 ax} = \frac{4\Gamma(\mu)\zeta(\mu-1)}{(2a)^\mu}.$$

The special value $\mu = 2m + 1$ and the identity

$$\zeta(2m) = \frac{(2\pi)^{2m}}{2(2m)!} |B_{2m}|,$$

give the result.