PROOF OF FORMULA 3.527.9

$$\int_0^\infty x^{2m+1}\frac{\cosh ax\,dx}{\sinh^2 ax} = \frac{2^{2m+1}-1}{a^2(2a)^{2m}}(2m+1)!\zeta(2m+1)$$

The result follows by putting $\mu = 2m+2$ in the formula

$$\int_0^\infty x^{\mu-1}\frac{\cosh ax\,dx}{\sinh^2 ax}\,dx = \frac{2\Gamma(\mu)\zeta(\mu-1)}{a^\mu}\left(1-2^{1-\mu}\right).$$

The change of variables $t = ax$ shows that it is sufficient to consider the special case $a = 1$. To establish this, write the hyperbolic functions as exponentials to produce

$$\int_0^\infty x^{\mu-1}\frac{\cosh x\,dx}{\sinh^2 x}\,dx = 2\int_0^\infty \frac{x^{\mu-1}(e^x+e^{-x})e^{-2x}\,dx}{(1-e^{-2x})^2}.$$

Expand the integrand as a series using

$$\sum_{k=1}^\infty ku^k = \frac{u}{(1-u)^2}$$

to obtain

$$\int_0^\infty x^{\mu-1}\frac{\cosh x\,dx}{\sinh^2 x}\,dx = 2\sum_{k=1}^\infty k\int_0^\infty x^{\mu-1}e^{-(2k-1)x}\,dx + 2\sum_{k=1}^\infty k\int_0^\infty x^{\mu-1}e^{-(2k+1)x}\,dx.$$

Scale the exponents of the integrals and the result follows.