PROOF OF FORMULA 3.541.3

$$\int_{-\infty}^{\infty} e^{-\mu x} \frac{\sinh \mu x}{\sinh \beta x} \, dx = \frac{\pi}{2\beta} \tan \left(\frac{\pi \mu}{\beta} \right)$$

The change of variables $t = \beta x$ shows that the integral is equivalent to

$$\int_{-\infty}^{\infty} e^{-at} \frac{\sinh at}{\sinh t} \, dt = \frac{\pi}{2} \tan \pi a.$$

In order to prove this, let $y = \ln t$ so that

$$\int_{-\infty}^{\infty} e^{-at} \frac{\sinh at}{\sinh t} \, dt = \int_{0}^{\infty} \frac{1 - y^{-2a}}{y^2 - 1} \, dy.$$

The change of variables $s = y^2$ gives

$$\int_{-\infty}^{\infty} e^{-at} \frac{\sinh at}{\sinh t} \, dt = \int_{0}^{\infty} \frac{s^{-1/2} - s^{-a-1/2}}{s - 1} \, ds.$$

The value is now obtained from entry **3.222.2** that states

$$\int_{0}^{\infty} \frac{x^{\mu-1} \, dx}{x + c} = \frac{\pi c^{\mu-1}}{\sin \pi \mu} \text{ for } c > 0$$

and

$$\int_{0}^{\infty} \frac{x^{\mu-1} \, dx}{x + c} = -\pi \cot(\pi \mu)(-c)^{\mu-1} \text{ for } c < 0.$$