PROOF OF FORMULA 3.541.5

$$\int_0^\infty \frac{e^{-px}dx}{(\cosh px)^{2q+1}} = \frac{2^{2q-2}}{p}B(q,q) - \frac{1}{2qp}$$

Make the change of variables $t = px$ to realize that it suffices to prove the formula for $p = 1$; that is,

$$\int_0^\infty \frac{e^{-t}dt}{(\cosh t)^{2q+1}} = 2^{2q-2}B(q,q) - \frac{1}{2q}.$$

The change of variables $y = \cosh t$ gives

$$e^t = y + \sqrt{y^2-1} \text{ and } e^{-t} = y - \sqrt{y^2-1}.$$

Taking derivatives and adding gives

$$dt = \frac{dy}{\sqrt{y^2-1}}.$$

This gives

$$\int_0^\infty \frac{e^{-t}dt}{(\cosh t)^{2q+1}} = \int_1^\infty \frac{dy}{y^{2q}\sqrt{y^2-1}} - \int_1^\infty \frac{dy}{y^{2q+1}}.$$

Since the value of the second integral is $1/(2q)$, it remains to prove that

$$\int_1^\infty \frac{dy}{y^{2q}\sqrt{y^2-1}} = 2^{2q-2}B(q,q).$$

The change of variables $s = y^2$ gives

$$\int_1^\infty \frac{dy}{y^{2q}\sqrt{y^2-1}} = \frac{1}{2}\int_1^\infty \frac{s^{-q-1/2}}{\sqrt{s-1}}\,ds$$

and then $u = 1/s$ yields

$$\int_1^\infty \frac{dy}{y^{2q}\sqrt{y^2-1}} = \frac{1}{2}\int_0^1 (1-u)^{-1/2}u^q\,dq.$$

This last integral is $B(q,\frac{1}{2})$ and this can be transformed using the identity

$$\frac{1}{\Gamma(q+\frac{1}{2})} = \frac{\Gamma(q)2^{2q-1}}{\sqrt{\pi}\Gamma(2q)}$$

that is the duplication formula for the gamma function, given as entry **8.335.1**.