PROOF OF FORMULA 3.562.4

$$\int_0^\infty xe^{-\beta x^2} \cosh \gamma x \, dx = \frac{\gamma}{4\beta}\sqrt{\frac{\pi}{\beta}} e^{\gamma^2/4\beta} \operatorname{erf}\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{1}{2\beta}$$

Write the integral as

$$\begin{aligned}
\int_0^\infty xe^{-\beta x^2} \cosh \gamma x \, dx &= \frac{1}{2}\int_0^\infty xe^{-\beta x^2+\gamma x} \, dx + \frac{1}{2}\int_0^\infty xe^{-\beta x^2-\gamma x} \, dx \\
&\quad + \frac{1}{2}e^{\gamma^2/4\beta}\int_0^\infty xe^{-\beta(x-\gamma/2\beta)^2} \, dx + \frac{1}{2}e^{\gamma^2/4\beta}\int_0^\infty xe^{-\beta(x+\gamma/2\beta)^2} \, dx \\
&\quad + \frac{1}{2}e^{\gamma^2/4\beta}\int_{-\gamma/2\beta}^\infty (t+\gamma/2\beta)e^{-\beta t^2} \, dt + \frac{1}{2}e^{\gamma^2/4\beta}\int_{\gamma/2\beta}^\infty (t-\gamma/2\beta)e^{-\beta t^2} \, dt.
\end{aligned}$$

Now let $s = \sqrt{\beta}t$ to obtain

$$\begin{aligned}
\int_0^\infty xe^{-\beta x^2} \cosh \gamma x \, dx &= \frac{1}{4\beta}e^{\gamma^2/4\beta}\int_{-\gamma/2\sqrt{\beta}}^\infty 2se^{-s^2} \, ds + \frac{\gamma}{4\beta\sqrt{\beta}}e^{\gamma^2/4\beta}\int_{-\gamma/2\sqrt{\beta}}^\infty e^{-s^2} \, ds \\
&\quad + \frac{1}{4\beta}e^{\gamma^2/4\beta}\int_{\gamma/2\sqrt{\beta}}^\infty 2se^{-s^2} \, ds - \frac{\gamma}{4\beta\sqrt{\beta}}e^{\gamma^2/4\beta}\int_{\gamma/2\sqrt{\beta}}^\infty e^{-s^2} \, ds \\
&= \frac{1}{2\beta} + \frac{\gamma e^{\gamma^2/4\beta}}{4\beta\sqrt{\beta}}\frac{\sqrt{\pi}}{2}\left[\operatorname{erf}\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{1}{2}\right] \\
&\quad - \frac{\gamma e^{\gamma^2/4\beta}}{4\beta\sqrt{\beta}}\frac{\sqrt{\pi}}{2}\left[-\operatorname{erf}\left(\frac{\gamma}{2\sqrt{\beta}}\right) + \frac{1}{2}\right]
\end{aligned}$$

and this reduces to the stated result.