PROOF OF FORMULA 3.622.4

$$\int_0^{\pi/4} \tan^{2n+1} x \, dx = (-1)^n \frac{\ln 2}{2} + \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k}$$

Let

$$I_n = \int_0^{\pi/4} \tan^{2n+1} x \, dx.$$

Then

$$\begin{aligned} I_n &= \int_0^{\pi/4} \tan^{2n+1} x \, dx = \int_0^{\pi/4} \tan^{2n-2} x \left(\sec^2 x - 1 \right) \, dx \\ &= -I_{n-1} + \frac{1}{2n}. \end{aligned}$$

Therefore,

$$I_n + I_{n-1} = \frac{1}{2n},$$

and

$$I_0 = \int_0^{\pi/4} \tan x \, dx = \int_0^{\pi/4} \frac{d}{dx} \ln \sec x \, dx = \frac{1}{2} \ln 2.$$

The formula can now be checked directly by induction.

Note. The evaluation should be written as

$$\int_0^{\pi/4} \tan^{2n+1} x \, dx = \frac{(-1)^n}{2} \left(\ln 2 + \sum_{k=0}^{n-1} \frac{(-1)^k}{k} \right).$$