

PROOF OF FORMULA 3.624.5

$$\int_0^{\pi/4} \frac{\sin^{2\mu-2} x}{(\cos x)^{2\mu}} dx = 2^{1-2\mu} B(2\mu - 1, 1 - \mu)$$

Use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and the change of variable $t = \cos 2x$ to obtain

$$\int_0^{\pi/4} \frac{\sin^{2\mu-2} x}{(\cos x)^{2\mu}} dx = \frac{1}{2^\mu} \int_0^1 t^{-\mu}(1-t)^{\mu-3/2}(1+t)^{-1/2} dt.$$

The integral representation

$${}_2F_1[\alpha, \beta; \gamma; z] = \frac{1}{B(\beta, \gamma - \beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-1}(1-tz)^{-\alpha} dt$$

gives the value

$$\begin{aligned} \int_0^{\pi/4} \frac{\sin^{2\mu-2} x}{(\cos x)^{2\mu}} dx &= \frac{1}{2^\mu} B(1 - \mu, \mu - \frac{1}{2}) {}_2F_1\left[\frac{1}{2}, 1 - \mu; \frac{1}{2}; -1\right] \\ &= \frac{1}{2^\mu} B(1 - \mu, \mu - \frac{1}{2}) 2^{\mu-1}. \end{aligned}$$

This is the result using

$$\Gamma(x + \frac{1}{2}) = \frac{\sqrt{\pi} \Gamma(2x)}{\Gamma(x) 2^{2x-1}}.$$