PROOF OF FORMULA 3.661.1

$$\int_0^{2\pi} (a \sin x + b \cos x)^{2n+1} \, dx = 0$$

Let $u = 2\pi - x$ to obtain

$$\int_0^{2\pi} (a \sin x + b \cos x)^{2n+1} \, dx = \int_0^{2\pi} (-a \sin x + b \cos x)^{2n+1} \, dx.$$

Expand using the binomial theorem and add the two expression to obtain

$$2 \int_0^{2\pi} (a \sin x + b \cos x)^{2n+1} \, dx = 2 \int_0^{2\pi} \sum_{j=0}^{n} \binom{2n+1}{2j} a^{2j} b^{2n+1-2j} \sin^{2j} x \cos^{2n+1-2j} x \, dx.$$

Define

$$J := \int_0^{2\pi} \sin^{2j} x \cos^{2n+1-2j} x \, dx.$$

Using periodicity

$$J := 2 \int_0^{\pi} \sin^{2j} x \cos^{2n+1-2j} x \, dx.$$

Finally,

$$\int_{\pi/2}^{\pi} \sin^{2j} x \cos^{2n+1-2j} x \, dx = - \int_0^{\pi/2} \sin^{2j} x \cos^{2n+1-2j} x \, dx,$$

by the change $x \mapsto \pi - x$. Therefore $J = 0$.