PROOF OF FORMULA 4.223.2

$$\int_0^\infty \ln(1-e^{-x})\,dx = -\frac{\pi^2}{6}$$

Start with the expansion

$$\ln(1-t) = -\sum_{k=0}^\infty \frac{t^{k+1}k+1}{}$$

to obtain

$$\ln(1-e^{-x}) = -\sum_{k=0}^\infty \frac{e^{-(k+1)x}}{k+1}.$$

Integration produces

$$\int_0^\infty \ln(1-e^{-x})\,dx = -\sum_{k=0}^\infty \frac{1}{k+1}\int_0^\infty e^{-(k+1)x}\,dx.$$

The change of variables $t=(k+1)x$ gives the result

$$\int_0^\infty \ln(1-e^{-x})\,dx = -\sum_{k=0}^\infty \frac{1}{(k+1)^2} = -\frac{\pi^2}{6}.$$