PROOF OF FORMULA 4.231.10

$$\int_0^\infty \frac{\ln x\,dx}{a^2-b^2x^2} = -\frac{\pi^2}{4ab}$$

This has to be interpreted as a Cauchy principal value

$$\int_0^\infty \frac{\ln x\,dx}{a^2-b^2x^2} = \lim_{\epsilon\to 0}\left[\int_0^{a/b-\epsilon} \frac{\ln x\,dx}{a^2-b^2x^2} + \int_{a/b+\epsilon}^\infty \frac{\ln x\,dx}{a^2-b^2x^2}\right].$$

The change of variable $x = at/b$ gives

$$\int_0^\infty \frac{\ln x\,dx}{a^2-b^2x^2} = \frac{1}{ab}\lim_{\epsilon\to 0}\left[\int_0^{1-b\epsilon/a} \frac{\ln(bt/a)\,dt}{1-t^2} + \int_{1+b\epsilon/a}^\infty \frac{\ln(bt/a)\,dt}{1-t^2}\right].$$

Let $s = 1/t$ in the second integral to get

$$\int_0^\infty \frac{\ln x\,dx}{a^2-b^2x^2} = \frac{1}{ab}\lim_{\epsilon\to 0}\left[\int_0^{(a-b\epsilon)/a} \frac{\ln(bt/a)\,dt}{1-t^2} + \int_0^{a/(a+b\epsilon)} \frac{\ln(at/b)\,dt}{1-t^2}\right].$$

Each integral is evaluated by the method of partial fractions to obtain

$$\begin{aligned}\int_0^\infty \frac{\ln x\,dx}{a^2-b^2x^2} \quad = \quad & \frac{\ln(b/a)}{ab}\lim_{\epsilon\to 0}\left[\int_0^{(a-b\epsilon)/a} \frac{dt}{1-t^2} - \int_0^{a/(a+b\epsilon)} \frac{dt}{1-t^2}\right] + \\ & + \frac{1}{ab}\lim_{\epsilon\to 0}\left[\int_0^{(a-b\epsilon)/a} \frac{\ln t\,dt}{1-t^2} + \int_0^{a/(a+b\epsilon)} \frac{\ln t\,dt}{1-t^2}\right].\end{aligned}$$

The integrands containing $\ln t$ converge to

$$\frac{2}{ab}\int_0^1 \frac{\ln t\,dt}{1-t^2} = -\frac{\pi^2}{4ab},$$

using entry **4.231.13**. The other two terms integrate to

$$\frac{1}{2}\ln(2a-b\epsilon) - \frac{1}{2}\ln(2a+b\epsilon)$$

that vanishes as $\epsilon \to 0$.