PROOF OF FORMULA 4.231.17

$$\int_0^1 \ln x \, \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \, dx = -(n+1)\frac{\pi^2}{12} - \sum_{k=1}^{n} (n-k+1) \frac{(-1)^k}{k^2}$$

Use the expansion

$$\frac{1+(-1)^n x^{n+1}}{1+x} = \sum_{j=0}^{n} (-1)^j x^j$$

and the series for $1/(1+x)$ to obtain

$$\int_0^1 \ln x \, \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \, dx = \sum_{j=0}^{n} (-1)^j \sum_{m=0}^{\infty} (-1)^m \int_0^1 x^{j+m} \ln x \, dx.$$

The change of variable $u = -\ln x$ yields

$$\int_0^1 \ln x \, \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \, dx = -\sum_{j=0}^{n} (-1)^j \sum_{m=0}^{\infty} (-1)^m \int_0^{\infty} u e^{-(m+j+1)u} \, du.$$

The change of variable $v = (m+j-1)u$ produces

$$\int_0^1 \ln x \, \frac{1+(-1)^n x^{n+1}}{(1+x)^2} \, dx = -\sum_{j=0}^{n} (-1)^j \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+j+1)^2}.$$

The change of index $r = m+j+1$ gives the result.