PROOF OF FORMULA 4.231.2

$$\int_0^1 \frac{\ln x \, dx}{1-x} = -\frac{\pi^2}{6}$$

Let $t = -\ln x$ to obtain

$$\int_0^1 \frac{\ln x \, dx}{1-x} = -\int_0^\infty \frac{t \, dt}{e^t - 1}.$$

Now use the expansion

$$\frac{1}{e^t - 1} = \frac{e^{-t}}{1 - e^{-t}} = e^{-t} \sum_{k=0}^\infty e^{-kt} = \sum_{k=1}^\infty e^{-kt}$$

to obtain

$$-\int_0^\infty \frac{t \, dt}{e^t - 1} = -\sum_{k=1}^\infty \int_0^\infty te^{-kt} \, dt.$$

The change of variables $x = kt$ and integration by parts gives

$$\int_0^\infty te^{-kt} \, dt = \frac{1}{k^2}.$$

This gives

$$\int_0^1 \frac{\ln x \, dx}{1-x} = -\sum_{k=1}^\infty \frac{1}{k^2} = -\frac{\pi^2}{6}.$$