PROOF OF FORMULA 4.231.7

$$\int_0^\infty \frac{\ln x\, dx}{(a^2+b^2x^2)^n} = \frac{\Gamma(n-\frac{1}{2})\sqrt{\pi}}{4(n-1)!\, a^{2n-1}b}\left[2\ln\left(\frac{a}{2b}\right) - \gamma - \psi\left(n-\frac{1}{2}\right)\right]$$

Let $t = bx/a$ to obtain

$$\int_0^\infty \frac{\ln x\, dx}{(a^2+b^2x^2)^n} = \frac{\ln(a/b)}{a^{2n-1}\, b} I_n + \frac{1}{a^{2n-1}\, b} J_n,$$

where

$$I_n = \int_0^\infty \frac{dt}{(1+t^2)^n} \text{ and } J_n = \int_0^\infty \frac{\ln t\, dt}{(1+t^2)^n}.$$

The change of variables $u = t^2$ gives

$$I_n = \frac{1}{2}\int_0^\infty \frac{u^{-1/2}\, du}{(1+u)^n} = \frac{1}{2}B\left(\frac{1}{2}, n-\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(n-\frac{1}{2})}{2(\,(n-1)!}.$$

The integral J_n is $f'(0)$, where

$$f(r) = \int_0^\infty \frac{t^r\, dt}{(1+t^2)^n}.$$

The change of variables $u = t^2$ gives

$$f(r) = \frac{1}{2}\int_0^\infty \frac{u^{r/2-1/2}\, du}{(1+u)^n} = \frac{\Gamma(r/2+1/2)\,\Gamma(n-r/2-1/2)}{2\,(n-1)!}.$$

Logarithmic differentiation gives

$$f'(0) = \frac{\Gamma(n-\frac{1}{2})\sqrt{\pi}}{4\,(n-1)!}\left[\psi\left(\tfrac{1}{2}\right) - \psi\left(n-\tfrac{1}{2}\right)\right].$$

The special values

$$\psi(n-\tfrac{1}{2}) = -\gamma - 2\ln 2 + 2H_{2n-2} - H_{n-1} \text{ and } \Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{(2m)!}2^{2m}m!$$

give the result.