PROOF OF FORMULA 4.241.5

$$\int_0^1 \sqrt{(1-x^2)^{2n-1}} \, \ln x \, dx = -\frac{(2n-1)!!}{(2n)!!} \frac{\pi}{4} \left(2 \ln 2 + \sum_{k=1}^{n} \frac{1}{k} \right)$$

The change of variable $t = x^2$ gives

$$\int_0^1 \sqrt{(1-x^2)^{2n-1}} \, \ln x \, dx = \frac{1}{4} \int_0^1 t^{-1/2}(1-t)^{n-1/2} \, \ln t \, dt.$$

In the proof of formula **4.253.1** it was shown that

$$\int_0^1 t^{a-1}(1-t)^{b-1} \ln t \, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \left[\psi(a) - \psi(a+b) \right].$$

Therefore

$$\int_0^1 \sqrt{(1-x^2)^{2n-1}} \, \ln x \, dx = \frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{4\Gamma(n+1)} \left[\psi(\tfrac{1}{2}) - \psi(n+1) \right].$$

The relations

$$\begin{aligned} \psi(\tfrac{1}{2}) &= -\gamma - 2\ln 2 \\ \psi(n+1) &= -\gamma + \sum_{k=1}^{n} \frac{1}{k} \end{aligned}$$

give the result.