PROOF OF FORMULA 4.251.2

$$\int_0^\infty \frac{x^{\mu-1}\,\ln x}{a-x}\,dx = \pi a^{\mu-1}\left(\cot\pi\mu\,\ln a - \frac{\pi}{\sin^2\pi\mu}\right)$$

The change of variables $x = at$ yields

$$\int_0^\infty \frac{x^{\mu-1}\,\ln x}{a-x}\,dx = a^{\mu-1}\ln a\int_0^\infty \frac{t^{\mu-1}\,dt}{1-t} + a^{\mu-1}\int_0^\infty \frac{t^{\mu-1}\,\ln t}{1-t}\,dt.$$

Split the first integral at $t = 1$ and change t by $1/t$ in the range $t \geq 1$. Then

$$\int_0^\infty \frac{t^{\mu-1}\,dt}{1-t} = \int_0^1 \frac{t^{\mu-1} - t^{-\mu}}{1-t}\,dt.$$

This integral has the value $\pi\cot\pi\mu$ according to entry **3.231.1**. The same splitting in the second integral gives

$$\int_0^\infty \frac{t^{\mu-1}\,\ln t}{1-t}\,dt = \int_0^1 \frac{t^{\mu-1}\ln t}{1-t}\,dt + \int_0^1 \frac{t^{-\mu}\ln t}{1-t}\,dt.$$

The result now follows from

$$\int_0^\infty \frac{t^{\mu-1}\,\ln t}{1-t}\,dt = -\psi'(\mu)$$

and the identity

$$\psi'(\mu) + \psi'(1-\mu) = \frac{\pi^2}{\sin^2\pi\mu}.$$