PROOF OF FORMULA 4.251.4

$$\int_0^1 \frac{x^{\mu-1}\,\ln x}{1-x}\,dx = -\psi'(\mu) = -\zeta(2,\mu)$$

The change of variables $t = -\ln x$ gives

$$\int_0^1 \frac{x^{\mu-1}\,\ln x}{1-x}\,dx = -\int_0^\infty \frac{te^{-\mu t}}{1-e^{-t}}\,dt$$

Entry **3.411.7** states that

$$\int_0^\infty \frac{x^{\nu-1}\,e^{-\mu x}}{1-e^{-bx}}\,dx = \frac{\Gamma(\nu)}{b^\nu}\zeta(\nu,\mu/b),$$

and with $\nu = 2$ and $b = 1$ we obtain

$$\int_0^\infty \frac{x\,e^{-\mu x}}{1-e^{-x}}\,dx = \zeta(2,\mu).$$

The relation to the polygamma function ψ comes from the identity

$$\psi^{(n)}(x) = (-1)^{n+1} n! \zeta(n+1,x)$$

given as **8.363.8**.