# PROOF OF FORMULA 4.254.2

$$\int_0^\infty \frac{x^{p-1}\,\ln x}{1-x^q}\,dx = -\frac{\pi^2}{q^2\sin^2(\pi p/q)}$$

Let $t = x^q$ to obtain

$$\int_0^\infty \frac{x^{p-1}\,\ln x}{1-x^q}\,dx = \frac{1}{q^2}\int_0^\infty \frac{t^{p/q-1}\,\ln t}{1-t}\,dt.$$

In the integral over $[1,\infty)$ let $s = 1/t$ to obtain

$$\int_1^\infty \frac{t^{p/q-1}\,\ln t}{1-t}\,dt = \int_0^1 \frac{s^{p/q}\,\ln s}{1-s}\,ds.$$

It follows that

$$\int_0^\infty \frac{x^{p-1}\,\ln x}{1-x^q}\,dx = \frac{1}{q^2}\int_0^1 \frac{\ln t}{1-t}(t^{p/q-1} + t^{-p/q})\,dt.$$

The integral representation for the digamma function

$$\psi(a) = -\int_0^1 \left(\frac{1}{\ln t} + \frac{t^{a-1}}{1-t}\right)\,dt$$

produces

$$\psi'(a) = -\int_0^1 \frac{t^{a-1}\,\ln t}{1-t}\,dt.$$

Therefore

$$\begin{aligned}\int_0^\infty \frac{x^{p-1}\,\ln x}{1-x^q}\,dx &= \frac{1}{q^2}\int_0^1 \frac{\ln t}{1-t}(t^{p/q-1} + t^{-p/q})\,dt \\ &= -\frac{1}{q^2}\left[\psi'\left(\frac{p}{q}\right) + \psi'\left(1-\frac{p}{q}\right)\right].\end{aligned}$$

Differentiating the identity

$$\psi(a) - \psi(1-a) = -\pi\cot\pi a,$$

at $a = p/q$, gives the result.