PROOF OF FORMULA 4.254.5

$$\int_0^\infty \frac{x^{p-1}\, \ln x}{1+x^q}\, dx = -\frac{\pi^2}{q^2} \frac{\cos(\pi p/q)}{\sin^2(\pi p/q)}$$

The change of variable $t = x^q$ gives

$$\int_0^\infty \frac{x^{p-1}\, \ln x}{1+x^q}\, dx = \frac{1}{q^2} \int_0^\infty \frac{t^{p/q-1}\, \ln t}{1+t}\, dt.$$

The integral representation

$$B(a, 1-a) = \int_0^\infty \frac{t^{a-1}\, dt}{1+t} = \Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin \pi a}$$

is differentiated at $a = p/q$ to produce the result.