PROOF OF FORMULA 4.255.3

$$\int_0^\infty \frac{1-x^p}{1-x^2}\,\ln x\,dx = \left[\frac{\pi}{2}\tan\left(\frac{\pi p}{2}\right)\right]^2$$

The change of variable $t = x^2$ gives

$$\int_0^\infty \frac{1-x^p}{1-x^2}\,\ln x\,dx = \frac{1}{4}\int_0^\infty \frac{t^{-1/2}\,\ln t}{1-t}\,dt - \frac{1}{4}\int_0^\infty \frac{t^{p/2-1/2}\,\ln t}{1-t}\,dt.$$

Split the integrals at $t=1$ and make the change of variable $t \mapsto 1/t$ in the one from 1 to ∞. This gives

$$\int_0^\infty \frac{t^{-1/2}\,\ln t}{1-t}dt = 2\int_0^1 \frac{t^{-1/2}\,\ln t}{1-t}\,dt.$$

The second integral gives

$$\int_0^\infty \frac{t^{p/2-1/2}\,\ln t}{1-t}\,dt = \int_0^1 \frac{t^{p/2-1/2}\,\ln t}{1-t}\,dt + \int_0^1 \frac{t^{-p/2-1/2}\,\ln t}{1-t}\,dt.$$

These three integrals are evaluated using

$$\int_0^1 \frac{t^{a-1}\,\ln t}{1-t}\,dt = -\psi'(a)$$

to produce

$$-2\psi'\left(\tfrac{1}{2}\right) - \psi'\left(\tfrac{p+1}{2}\right) - \psi'\left(\tfrac{-p+1}{2}\right).$$

The result is simplified using

$$\psi'(a) + \psi'(1-a) = \frac{\pi^2}{\sin^2 \pi a}.$$