

PROOF OF FORMULA 4.261.15

$$\int_0^1 \ln^2 x \frac{x^{2n} dx}{\sqrt{1-x^2}} = \frac{(2n-1)!! \pi}{2(2n)!!} \left\{ \frac{\pi^2}{12} + \sum_{k=1}^{2n} \frac{(-1)^k}{k^2} + \left[\sum_{k=1}^{2n} \frac{(-1)^k}{k} + \ln 2 \right]^2 \right\}$$

The change of variables $t = x^2$ gives

$$\int_0^1 \ln^2 x \frac{x^{2n} dx}{\sqrt{1-x^2}} = \frac{1}{8} \int_0^1 \ln^2 t t^{n-1/2} (1-t)^{-1/2} dt.$$

Entry **4.261.17** states that

$$\int_0^1 \ln^2 x x^{\mu-1} (1-x)^{\nu-1} dx = B(\mu, \nu) [(\psi(\mu) - \psi(\mu + \nu))^2 + \psi'(\mu) - \psi'(\mu + \nu)]$$

gives

$$\int_0^1 \ln^2 x \frac{x^{2n} dx}{\sqrt{1-x^2}} = \frac{1}{8} B\left(n + \frac{1}{2}, \frac{1}{2}\right) [(\psi(n + 1/2) - \psi(n + 1))^2 + \psi'(n + 1/2) - \psi'(n + 1)].$$

Now use

$$B\left(n + \frac{1}{2}, \frac{1}{2}\right) = \frac{\pi (2n)!}{2^{2n} (n!)^2}$$

and

$$\begin{aligned} \psi(n+1) &= -\gamma + \sum_{k=1}^n \frac{1}{k} \\ \psi(n+1/2) &= -\gamma + 2 \left[\sum_{k=1}^n \frac{1}{2k-1} - \ln 2 \right] \\ \psi'(n+1) &= \frac{\pi^2}{6} - \sum_{k=1}^n \frac{1}{k^2} \\ \psi'(n+1/2) &= \frac{\pi^2}{2} - 4 \sum_{k=1}^n \frac{1}{(2k-1)^2} \end{aligned}$$

that appear as entries **8.365.4**, **8.366.3**, **8.366.11** and **8.366.12**, respectively, yield the result.