PROOF OF FORMULA 4.261.21

$$\int_0^1 x^{p-1}(1-x^r)^{q-1} \ln^2 x \, dx = \frac{B(p/r,q)}{r^3} \left(\psi'\left(\frac{p}{r}\right) - \psi'\left(\frac{p}{r}+q\right) + \left[\psi\left(\frac{p}{r}\right) - \psi\left(\frac{p}{r}+q\right)\right]^2 \right)$$

The change of variables $t = x^r$ gives

$$\int_0^1 x^{p-1}(1-x^r)^{q-1} \ln^2 x \, dx = \frac{1}{r^3} \int_0^1 t^{p/r-1}(1-t)^{q-1} \ln^2 t \, dt.$$

The result now follows from Entry **4.261.17** which states that

$$\int_0^1 x^{\mu-1}(1-x)^{\nu-1} \ln^2 x \, dx = B(\mu,\nu) \left[(\psi(\mu) - \psi(\mu+\nu))^2 + \psi'(\mu) - \psi'(\mu+\nu) \right].$$