PROOF OF FORMULA 4.262.1

$$\int_0^1 \frac{\ln^3 x}{1+x}\,dx = -\frac{7\pi^4}{120}$$

Expand the integrand to get

$$\int_0^1 \frac{\ln^3 x}{1+x}\,dx = \sum_{j=0}^{\infty}(-1)^j \int_0^1 x^j \ln^3 x\,dx.$$

The changes of variables $t = -\ln x$ and $s = (j+1)t$ produce

$$\int_0^1 \frac{\ln^3 x}{1+x}\,dx = -\sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)^4} \int_0^{\infty} s^3 e^{-s} ds.$$

The integral is recognized as $\Gamma(4) = 6$ to obtain

$$\int_0^1 \frac{\ln^3 x}{1+x}\,dx = 6\sum_{j=1}^{\infty} \frac{(-1)^j}{j^4}$$

The usual even-odd trick for the zeta function and $\zeta(4) = \pi^4/90$ give the final result.